# Existence of optimal controls for systems driven by FBSDEs☆

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## ABSTRACT

We prove the existence of optimal relaxed controls as well as strict optimal controls for systems governed by non linear forward–backward stochastic differential equations (FBSDEs). Our approach is based on weak convergence techniques for the associated FBSDEs in the Jakubowski S-topology and a suitable Skorokhod representation theorem.

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## 1. Introduction

In this paper, we study the existence of optimal controls for systems driven by FBSDEs of the form

$$\begin{cases} X_t = x + \int_0^t b(s, X_s, U_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}W_s, \\ Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, U_s)\mathrm{d}s \\ \quad - \int_t^T Z_s \mathrm{d}W_s - (M_T - M_t) \end{cases} \quad (1.1)$$

where $b, \sigma, f$ and $g$ are given functions, $(W_t, t \geq 0)$ is a standard Brownian motion, defined on some filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, satisfying the usual conditions. $X, Y, Z$ are square integrable adapted processes and $M$ is a square integrable martingale which is orthogonal to $W$. The control variable $U_t$, called strict control, is a measurable, $\mathcal{F}_t$-adapted process with values in a compact metric space $\mathbb{A}$. The expected cost on the time interval $[0, T]$ is of the form

$$J(U) = E\left[l(Y_0) + \int_0^T h(t, X_t, Y_t, U_t)\mathrm{d}t\right]. \quad (1.2)$$

The objective of the controller is to minimize this cost function, over the class $\mathbb{U}$ of admissible controls, that is, adapted processes with values in some set $\mathbb{A}$, called the action space. A control $\widehat{u}$ is called optimal if it satisfies $J(\widehat{u}) = \inf\{J(u), u \in \mathbb{U}\}$.

In the case of forward Itô's SDEs, the existence of such a strict optimal control follows from the convexity of the image of the action space $\mathbb{A}$ by the mapping $(b(t, x, .), \sigma^2(t, x, .), h(t, x, .))$, which is known as the Roxin-type convexity condition, see for instance [1–3]. Without this convexity condition, an optimal control may fail to exist in $\mathbb{U}$. It should be noted that the set $\mathbb{U}$ is not equipped with a compact topology. The idea is then to introduce a new class $\mathcal{R}$ of admissible controls, in which the controller chooses at time $t$, a probability measure $q_t(\mathrm{d}a)$ on the control set $\mathbb{A}$, rather than an element $u_t \in \mathbb{A}$. These are called relaxed controls.

Using compactification techniques, Fleming [4] derived the first existence result of an optimal relaxed control for SDEs with uncontrolled diffusion coefficient. The case of SDEs with a controlled diffusion coefficient has been solved by El-Karoui et al. [1], where the optimal relaxed control is shown to be Markovian.

Linear backward stochastic differential equations (BSDEs) have been studied in the early seventies by Bismut [5], in connection with the stochastic version of the Pontriagin maximum principle. More precisely, the adjoint process in the maximum principle satisfies a linear BSDE. The first existence and uniqueness result for non linear BSDEs has been proved by Pardoux and Peng [6]. This important paper has given rise to a huge literature on BSDEs and has become a powerful tool in many fields such as financial mathematics, optimal control, stochastic games, semi linear and quasi linear partial differential equations, differential geometry

and homogenization, see e.g. [7–10]. Therefore it becomes quite natural to investigate control problems for systems governed by BSDEs and FBSDEs. Stochastic control problems for systems driven by FBSDEs or BSDEs have been studied by many authors, see e.g. [11–18] and the references therein. These papers have been devoted to various forms of the stochastic maximum principle. The problem of existence of optimal controls for systems driven by BSDEs has been studied for the first time in [19]. The authors suppose that the generator is linear and assume convexity of the cost function as well as the action space. They show the existence of an optimal strong control, that is an optimal control adapted to the original filtration of the Brownian motion.

In this paper, we prove the existence of an optimal relaxed control for systems driven by non linear FBSDEs. The proof of the main result is based on tightness results of the distributions of the processes defining the control problem and the Skorokhod representation theorem on the space $\mathbb{D}$, endowed with the Jakubowski $S$-topology [20]. Furthermore, when the Roxin convexity condition is fulfilled, we prove that the optimal relaxed control is in fact strict.

The paper is organized as follows. In Section 2, we give some preliminaries and assumptions made on the model. In Section 3, we define precisely the relaxed optimal control problem and present the main result. Section 4 is devoted to the proofs.

## 2. Formulation of the problem, notations and assumptions

We consider an optimal control problem of a system driven by the FBSDE (1.1), where $U$ is a strict control, that is a measurable, $\mathcal{F}_t$-adapted process with values in a compact metric space $\mathbb{A}$. $(W_t, t \geq 0)$ is a $m$-dimensional Brownian motion defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ and $M$ is a square integrable martingale which is orthogonal to $W$. It should be noted that the probability space and the Brownian motion may change with the control $U$.

The idea of relaxed control consists to replace the $\mathbb{A}$-valued process $(U_t)$ with $P(\mathbb{A})$-valued process $(q_t)$, where $P(\mathbb{A})$ is the space of probability measures equipped with the topology of weak convergence. We denote by $\mathbb{V}$ the set of probability measures on $[0, T] \times \mathbb{A}$ whose projections on $[0, T]$ coincide with the Lebesgue measure $dt$. Equipped with the topology of stable convergence of measures, $\mathbb{V}$ is a compact metrizable space, see Jacod & Mémin [21]. Stable convergence is required for bounded measurable functions $h(t, a)$ such that for each fixed $t \in [0, T]$, $h(t, .)$ is continuous. A relaxed control could be identified as a random variable with values in $\mathbb{V}$ (see [8]). The system is then governed by the following FBSDE

$$\begin{cases} X_t = x + \int_0^t \int_A b(s, X_s, a) q_s(da) ds + \int_0^t \sigma(s, X_s) dW_s, \\ Y_t = g(X_T) + \int_t^T \int_A f(s, X_s, Y_s, a) q_s(da) ds \\ \quad - \int_t^T Z_s dW_s - (M_T - M_t) \end{cases} \quad (2.1)$$

where $M$ is a square integrable martingale which is orthogonal to $W$.

The cost to be minimized, over the set $\mathcal{R}$ of relaxed controls, has the form

$$J(q) = E\left(l(Y_0) + \int_0^T \int_A h(t, X_t, Y_t, a) q_t(da) dt\right). \quad (2.2)$$

We denote:

$$\mathcal{M}^2(t, T; \mathbb{R}^{d \times m}) := \left\{X : [0, T] \times \Omega \longrightarrow \mathbb{R}^{d \times m}, \right.$$

$$\left. X \text{ progressively measurable: } E \int_0^T |X_t|^2 dt < \infty \right\},$$

and

$$\mathcal{S}^2(t, T; \mathbb{R}^d) := \left\{X : [0, T] \times \Omega \longrightarrow \mathbb{R}^d, \right.$$

$$\left. X \text{ progressively measurable: } E\left(\sup_{0 \leq t \leq T} |X_t|^2\right) < \infty \right\}.$$

Under assumptions (**A1**) and (**A2**), the system (2.1) has a unique solution $(X_t, Y_t, Z_t) \in [\mathcal{S}^2(t, T; \mathbb{R}^d)]^2 \times [\mathcal{M}^2(t, T; \mathbb{R}^d)]$.

(**A1**) Assume that the functions

$b : [0, T] \times \mathbb{R}^d \times \mathbb{A} \to \mathbb{R}^d$,

$\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times m}$,

$f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{A} \to \mathbb{R}^k$

$g : \mathbb{R}^d \to \mathbb{R}^k$

are continuous. Moreover assume that there exist a constant $K_1 > 0$ such that for every $(t, x, y, u) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{A}$,

$|b(t, x, u)| + |\sigma(t, x)| + |g(x)| \leq K_1(1 + |x|)$

$|f(t, x, y, u)| \leq K_1(1 + |x| + |y|)$.

(**A2**) There exists a constant $K > 0$ such that for every $t \in [0, T]$, every $x, x' \in \mathbb{R}^d$ and every $y, y' \in \mathbb{R}^k$,

$|f(t, x, y, u) - f(t, x', y', u)| \leq K(|x - x'| + |y - y'|)$,

$|b(t, x, u) - b(t, x', u)| \leq K|x - x'|$,

$|\sigma(t, x) - \sigma(t, x')| \leq K|x - x'|$.

The cost corresponding to a control $U$ is defined by

$$J(U) := E\left(l(Y_0) + \int_0^T h(t, X_t, Y_t, U_t) dt\right). \quad (2.3)$$

(**A3**)

$h : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{A} \to \mathbb{R}$,

$l : \mathbb{R}^d \to \mathbb{R}$,

are continuous functions with a linear growth in $(x, y)$ uniformly in $(t, a)$. Moreover assume that $h := h(t, x, y, a)$ is Lipschitz in $(x, y)$ uniformly in $(t, a)$.

In the sequel we denote by:

$\mathcal{C}([0, T], \mathbb{R}^d)$: the space of continuous functions from $[0, T]$ into $\mathbb{R}^d$, equipped with the topology of uniform convergence.

$\mathbb{D}([0, T]; \mathbb{R}^k)$: the Skorokhod space of càdlàg functions from $[0, T]$ into $\mathbb{R}^d$, that is functions which are continuous from the right with left hand limits.

## 3. The main result

**Theorem 3.1.** *Under conditions* (**A1**)–(**A3**), *the relaxed control problem has an optimal solution.*

To deal with the existence of a strict optimal control, we need the Roxin condition, given by

(**A4**) (Roxin's condition). For every $(t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^k$, the set

$(b, f, h)(t, x, y, \mathbb{A}) := \{b_i(t, x, u), f_j(t, x, y, u),$

$\quad h(t, x, y, u) \setminus u \in \mathbb{A}, i = 1, \ldots, d, j = 1, \ldots, k\}$,

is convex and closed in $\mathbb{R}^{d+k+1}$.

**Corollary 3.2.** *Assume that* (**A1**)–(**A4**) *hold. Then, the relaxed optimal control* $\hat{q}_t$ *has the form of a Dirac measure charging a strict control* $\hat{U}_t$ *(i.e.; $\hat{q}_t(da) = \delta_{\hat{U}_t}(da)$).*

## 4. Proof of the main results

To prove Theorem 3.1, we need some auxiliary results on the tightness of the processes under consideration.

Let $(q^n)_{n\geq 0}$ be a minimizing sequence, that is $\lim_{n\to\infty} J(q^n) = \inf_{\mu\in\mathcal{R}} J(\mu)$. Let $(X^n, Y^n, Z^n)$ be the unique solution of the FBSDE

$$\begin{cases} X_t^n = x + \int_0^t \int_A b(s, X_s^n, a)q_s^n(da)ds + \int_0^t \sigma(s, X_s^n)dW_s, \\ Y_t^n = g(X_T^n) + \int_t^T \int_A f(s, X_s^n, Y_s^n, a)q_s^n(da)ds - \int_t^T Z_s^n dW_s. \end{cases} \quad (4.1)$$

**Lemma 3.3.** *Let $(X^n, Y^n, Z^n)$ be the unique solution of Eq. (2.1). There exists a positive constant $C$ such that*

$$\sup_n E\left(\sup_{0\leq t\leq T} |X_t^n|^2 + \sup_{0\leq t\leq T} |Y_t^n|^2 + \int_t^T |Z_s^n|^2 ds\right) \leq C. \quad (4.2)$$

**Proof.** Let $(q^n)_{n\geq 0}$ be a minimizing sequence (i.e., $\lim_{n\to\infty} J(q^n) = \inf_{\mu\in\mathcal{R}} J(\mu)$). Using assumption (**A1**), it is easy to check that

$$\sup_n \left\{E\left(\sup_{0\leq t\leq T} |X_t^n|^2\right)\right\} < \infty. \quad (4.3)$$

Using the Burkholder–Davis–Gundy inequality and the Schwarz inequality, one can show that the local martingale $\int_t^T Y_s^n Z_s^n dW_s$ is a uniformly integrable martingale. It then follows by using Itô's formula and assumption (**A1**) that

$$E\left(|Y_t^n|^2 + \int_t^T |Z_s^n|^2 ds\right)$$
$$= E\left(|g(X_T^n)|^2 + 2\int_t^T \int_A \langle Y_s^n, f(s, X_s^n, Y_s^n, a)\rangle q_s^n(da)ds\right).$$

Hence,

$$E\left(|Y_t^n|^2 + \int_t^T |Z_s^n|^2 ds\right) \leq E\left(|g(X_T^n)|^2 + \int_t^T |Y_s^n|^2 ds\right)$$
$$+ E\left(\int_t^T \int_A |f(s, X_s^n, Y_s^n, a)|^2 q_s^n(da)ds\right).$$

Now, Gronwall's lemma allows us to show that

$$\sup_n E\left(\left(\sup_{0\leq t\leq T} |Y_t^n|^2 + \int_t^T |Z_s^n|^2 ds\right)\right) < \infty. \quad \square$$

**Lemma 3.4.** *Let $(X^n, Y^n, Z^n)$ be the unique solution of Eq. (2.1). The sequence $(Y^n, \int_0^{\cdot} Z_s^n dW_s)$ is tight on the space $\mathbb{D}([0, T]; \mathbb{R}^k) \times \mathbb{D}([0, T]; \mathbb{R}^k)$ endowed with the S-topology.*

**Proof.** Let $0 = t_0 < t_1 < \cdots < t_n = T$. We define the conditional variation by

$$CV(Y^n) := \sup E\left[\sum_i |E(Y_{t_{i+1}}^n - Y_{t_i}^n)/\mathcal{F}_{t_i}^W|\right]$$

where the supremum is taken over all partitions of the interval $[0, T]$. It is proved in [10] that

$$CV(Y^n) \leq \mathbb{E}\left[\int_0^T \int_A |f(s, X_s^n, Y_s^n, a)|q_s^n(da)ds\right].$$

It follows from (4.2) that

$$\sup_n \left[CV(Y^n) + \sup_{0\leq t\leq T} E|Y_t^n| + \sup_{0\leq t\leq T} E\left|\int_0^t Z_s^n dW_s\right|\right] < \infty.$$

Therefore, the sequences $Y^n$ and $M^n := \int_0^{\cdot} Z_s^n dW_s$ satisfy the Meyer & Zheng tightness criterion [22]. $\square$

**Lemma 3.5.** *The family of relaxed controls $(q^n)_{n\geq 0}$ is tight in $\mathbb{V}$.*

**Proof.** $[0, T] \times \mathbb{A}$ being compact, then by Prokhorov's theorem, the space $\mathbb{V}$ of probability measures on $[0, T] \times \mathbb{A}$ is then compact for the topology of weak convergence. The fact that $q^n$, $n \geq 0$ is a random variable with values in the compact set $\mathbb{V}$ implies that the family of distributions associated to $(q^n)_{n\geq 0}$ is tight. $\square$

The next lemma may be proved by standard arguments.

**Lemma 3.6.** *Let $X_t^n$ be the forward component of Eq. (2.1). Then, the sequence of processes $(X^n, W)$ is tight on the space $\mathcal{C}([0, T], \mathbb{R}^d) \times \mathcal{C}([0, T], \mathbb{R}^m)$, endowed with the topology of uniform convergence.*

### 4.1. Proof of Theorem 3.1

Let $(q^n)_{n\geq 0}$ be a minimizing sequence, that is $\lim_{n\to\infty} J(q^n) = \inf_{\mu\in\mathcal{R}} J(\mu)$. Let $(X^n, Y^n, Z^n)$ be the unique solution of the FBSDE

$$\begin{cases} X_t^n = x + \int_0^t \int_A b(s, X_s^n, a)q_s^n(da)ds + \int_0^t \sigma(s, X_s^n)dW_s, \\ Y_t^n = g(X_T^n) + \int_t^T \int_A f(s, X_s^n, Y_s^n, a)q_s^n(da)ds - (M_T^n - M_t^n) \end{cases} \quad (4.4)$$

where $M_t^n := \int_0^t Z_s^n dW_s$.

From Lemmas 3.4–3.6, it follows that the sequence of processes

$$\gamma^n = (X^n, W, q^n, Y^n, M^n)$$

is tight on the space

$$\Gamma = \mathcal{C}([0, T], \mathbb{R}^d) \times \mathcal{C}([0, T], \mathbb{R}^m) \times \mathbb{V} \times [\mathbb{D}([0, T]; \mathbb{R}^k)]^2$$

equipped with the product topology of the uniform convergence on the first factor, the topology of stable convergence of measures on the second factor and the S-topology on the third factor. By Jakubowski [20] (see the Appendix), there exists a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$, a sequence $\hat{\gamma}^n = (\hat{X}^n, \hat{W}^n, \hat{q}^n, \hat{Y}^n, \hat{M}^n)$ and $\hat{\gamma} = (\hat{X}, \hat{W}, \hat{q}, \hat{Y}, \hat{M})$ defined on this space such that:

(i) for each $n \in \mathbb{N}$, law$(\gamma^n) =$ law$(\hat{\gamma}^n)$,
(ii) there exists a subsequence $(\hat{\gamma}^{n_k})$ of $(\hat{\gamma}^n)$, still denoted $(\hat{\gamma}^n)$, which converges to $\hat{\gamma}$, $P$-a.s. on the space $\Gamma$,
(iii) $(\hat{Y}^n, \hat{M}^n)$ converges to $(\hat{Y}, \hat{M})$, $dt \times \hat{P}$-a.s., and $(\hat{Y}_T^n, \hat{M}_T^n)$ converges to $(\hat{Y}_T, \hat{M}_T)$ as $n \to \infty$, $\hat{P}$-a.s.
(iv) $\sup_{0\leq t\leq T} |\hat{X}_t^n - \hat{X}_t| \to 0$, $\hat{P}$-a.s.

According to property (i), we get

$$\begin{cases} \hat{X}_t^n = x + \int_0^t \int_A b(s, \hat{X}_s^n, a)\hat{q}_s^n(da)ds + \int_0^t \sigma(s, \hat{X}_s^n)d\hat{W}_s^n \\ \hat{Y}_t^n = g(\hat{X}_T^n) + \int_t^T \int_A f(s, \hat{X}_s^n, \hat{Y}_s^n, a)\hat{q}_s^n(da)ds - (\hat{M}_T^n - \hat{M}_t^n) \end{cases} \quad (4.5)$$

where $\hat{M}_t^n := \int_t^T \hat{Z}_s^n d\hat{W}_s^n$.

Using properties (ii)–(iv), assumptions (**A1**)–(**A2**), then passing to the limit in the FBSDE (4.5), one can show that there exists a countable set $D \subset [0, T)$ such that

$$\begin{cases} \hat{X}_t = x + \int_0^t \int_A b(s, \hat{X}_s, a)\hat{q}_s(da)ds \\ \quad + \int_0^t \sigma(s, \hat{X}_s)d\hat{W}_s, \quad t > 0, \\ \hat{Y}_t = g(\hat{X}_T) + \int_t^T \int_A f(s, \hat{X}_s, \hat{Y}_s, a)\hat{q}_s(da)ds \\ \quad - (\hat{M}_T - \hat{M}_t), \quad t \in [0, T] \setminus D. \end{cases} \quad (4.6)$$

Since $\hat{Y}$ and $\hat{M}$ are càdlàg, it follows that for every $t \in [0, T]$

$$\hat{Y}_t = g(\hat{X}_T) + \int_t^T \int_A f(s, \hat{X}_s, \hat{Y}_s, a)\hat{q}_s(da)ds + \hat{M}_t - \hat{M}_T.$$

Since all the previous identifications of the limits (from Eq. (4.5) to Eq. (4.6)) can be proved by using the same arguments, we only explain how the following limit holds in probability

$$\lim_{n \to \infty} \int_t^T \int_A f(s, \hat{X}_s^n, \hat{Y}_s^n, a)\hat{q}_s^n(da)ds$$
$$= \int_t^T \int_A f(s, \hat{X}_s, \hat{Y}_s, a)\hat{q}_s(da)ds. \tag{4.7}$$

We use properties (i), (ii), (iv), Fatou's lemma and Lemma 3.3, to show that there exists a positive constant $C$ such that:

$$\hat{E}\left(\int_0^T (|\hat{X}_s|^2 + |\hat{Y}_s|^2)ds\right) \le C. \tag{4.8}$$

On the other hand, we have

$$\left| \int_t^T \int_A f(s, \hat{X}_s^n, \hat{Y}_s^n, a)\hat{q}_s^n(da)ds \right.$$
$$\left. - \int_t^T \int_A f(s, \hat{X}_s, \hat{Y}_s, a)\hat{q}_s(da)ds \right| = I(n) + J(n)$$

where

$$I(n) := \left| \int_t^T \int_A f(s, \hat{X}_s^n, \hat{Y}_s^n, a)\hat{q}_s^n(da)ds \right.$$
$$\left. - \int_t^T \int_A f(s, \hat{X}_s, \hat{Y}_s, a)\hat{q}_s^n(da)ds \right|$$

$$J(n) := \left| \int_t^T \int_A f(s, \hat{X}_s, \hat{Y}_s, a)\hat{q}_s^n(da)ds \right.$$
$$\left. - \int_t^T \int_A f(s, \hat{X}_s, \hat{Y}_s, a)\hat{q}_s(da)ds \right|.$$

Let us prove that $I(n)$ converges to 0 in probability. Let $\varepsilon > 0$. We use assumption (**A2**) to obtain,

$$\hat{P}\left\{ \left| \int_t^T \int_A f(s, \hat{X}_s^n, \hat{Y}_s^n, a)\hat{q}_s^n(da)ds \right. \right.$$
$$\left. \left. - \int_t^T \int_A f(s, \hat{X}_s, \hat{Y}_s, a)\hat{q}_s^n(da)ds \right| > \varepsilon \right\}$$
$$\le \frac{1}{\varepsilon}\hat{E}\int_t^T |f(s, \hat{X}_s^n, \hat{Y}_s^n, a) - f(s, \hat{X}_s, \hat{Y}_s, a)|ds$$
$$\le \frac{K}{\varepsilon}\left[ \hat{E}\int_t^T |\hat{X}_s^n - \hat{X}_s|ds + \hat{E}\int_t^T |\hat{Y}_s^n - \hat{Y}_s|ds \right].$$

Now, properties (i)–(iv) and Lemma 3.3 allow us to show that $\hat{E}\int_t^T |\hat{X}_s^n - \hat{X}_s|ds + \hat{E}\int_t^T |\hat{Y}_s^n - \hat{Y}_s|ds$ tends to 0 as $n$ tends to infinity, which yields that $I(n)$ converges to 0 in probability.

We shall prove that $J(n)$ converges to 0 in probability. Let $R > 0$ and, put $B := \{|\hat{X}_s| + |\hat{Y}_s| \le R\}$ and $\bar{B} := \Omega - B$. We have,

$$\left| \int_t^T \int_A f(s, \hat{X}_s, \hat{Y}_s, a)\hat{q}_s^n(da)ds \right.$$
$$\left. - \int_t^T \int_A f(s, \hat{X}_s, \hat{Y}_s, a)\hat{q}_s(da)ds \right| = I_1(n) + J_1(n)$$

where

$$I_1(n) := \left| \int_t^T \int_A f(s, \hat{X}_s, \hat{Y}_s, a)\mathbb{1}_B \hat{q}_s^n(da)ds \right.$$
$$\left. - \int_t^T \int_A f(s, \hat{X}_s, \hat{Y}_s, a)\mathbb{1}_B \hat{q}_s(da)ds \right|$$

$$J_1(n) := \left| \int_t^T \int_A f(s, \hat{X}_s, \hat{Y}_s, a)\mathbb{1}_{\bar{B}} \hat{q}_s^n(da)ds \right.$$
$$\left. - \int_t^T \int_A f(s, \hat{X}_s, \hat{Y}_s, a)\mathbb{1}_{\bar{B}} \hat{q}_s(da)ds \right|.$$

Since the function $(s, a) \longmapsto f(s, \hat{X}_s, \hat{Y}_s, a)\mathbb{1}_B$ is bounded measurable in $(s, a)$ and continuous in $a$, we deduce by using property (ii) that $I_1(n)$ tends to 0 in probability as $n$ tends to $\infty$. It remains to prove that $J_1(n)$ tends to 0 in probability as $n$ tends to $\infty$. We have,

$$\hat{E}[J_1(n)] = \hat{E}\left( \left| \int_t^T \int_A f(s, \hat{X}_s, \hat{Y}_s, a)\mathbb{1}_{\bar{B}} \hat{q}_s^n(da)ds \right. \right.$$
$$\left. \left. - f(s, \hat{X}_s, \hat{Y}_s, a)\mathbb{1}_{\bar{B}} \hat{q}_s(da)ds \right| \right)$$
$$\le \hat{E}\int_t^T |f(s, \hat{X}_s, \hat{Y}_s, a)|\mathbb{1}_{\bar{B}}ds + \hat{E}\int_t^T |f(s, \hat{X}_s, \hat{Y}_s, a)|\mathbb{1}_{\bar{B}}ds$$
$$\le \frac{K'}{R^2}\hat{E}\int_t^T (|\hat{X}_s|^2 + |\hat{Y}_s|^2)ds.$$

We successively pass to the limit in $n$ and $R$, to show that $\lim_{n \to \infty} J_1(n) = 0$ in probability. (4.7) is proved.

Now, let $\hat{\mathcal{F}}_s := \mathcal{F}_s^{\hat{X}, \hat{Y}, \hat{q}}$, be the minimal admissible and complete filtration generated by $(\hat{X}_r, \hat{Y}_r, \hat{q}_r, \ r \le s)$. Combining the estimates (4.2), Lemmas 4 and 5 in Appendix, we show that $\hat{M}$ is a $\hat{\mathcal{F}}_s$-martingale. Therefore by the martingale decomposition theorem, there exists a process $\hat{Z} \in \mathcal{M}^2(t, T; \mathbb{R}^{n \times d})$ such that

$$\hat{M}_t = \int_0^t \hat{Z}_s d\hat{W}_s + \hat{N}_t, \quad \text{with } \langle \hat{N}, \hat{W} \rangle_t = 0,$$

which implies that

$$\hat{Y}_t = g(\hat{X}_T) + \int_t^T \int_A f(s, \hat{X}_s, \hat{Y}_s, a)\hat{q}_s(da)ds$$
$$- \int_t^T \hat{Z}_s d\hat{W}_s - (\hat{N}_T - \hat{N}_t).$$

To finish the proof of Theorem 3.1, it remains check that $\hat{q}$ is an optimal control.

According to above properties (i)–(iv) and assumption (**A3**), we have

$$\inf_{q \in \mathcal{R}} J(q) = \lim_{n \to \infty} J(q^n),$$
$$= \lim_{n \to \infty} E\left[ l(Y_0^n) + \int_0^T \int_A h(t, X_t^n, Y_t^n, a)q_t^n(da)dt \right]$$
$$= \lim_{n \to \infty} \hat{E}\left[ l(\hat{Y}_0^n) + \int_0^T \int_A h(t, \hat{X}_t^n, \hat{Y}_t^n, a)\hat{q}_t^n(da)dt \right]$$
$$= \hat{E}\left[ l(\hat{Y}_0) + \int_0^T \int_A h(t, \hat{X}_t, \hat{Y}_t, a)\hat{q}_t(da)dt \right].$$

Theorem 3.1 is proved. $\quad\square$

### 4.2. Proof of Corollary 3.2

We put

$$\int_A f(s, \hat{X}_s, \hat{Y}_s, a)\hat{q}_s(da) := \hat{f}(t, w) \in f(t, x, y, \mathbb{U}),$$

$$\int_A h(t, \hat{X}_t, \hat{Y}_t, a)\hat{q}_t(da) := \hat{h}(t, w) \in h(t, x, y, \mathbb{U})$$

$$\int_A b(s, \hat{X}_s^n, a)\hat{q}_s(da) := \widehat{b}(t, w) \in b(t, x, \mathbb{U}).$$

From (**A4**) and the measurable selection theorem (see [23] p. 74), there is a $\mathbb{A}$-valued, $\mathcal{F}^{\hat{X}, \hat{Y}, \hat{q}}$-adapted process $\hat{U}$, such that for every $s \in [0, T] \setminus D$ and $w \in \hat{\Omega}$,

$(\hat{f}, \hat{h})(s, w) = (f, h)(s, \hat{X}(s, w), \hat{Y}(s, w), \hat{U}(s, w)),$

$\hat{b}(s, w) = b(s, \hat{X}(s, w), \hat{U}(s, w)).$

Hence, for every $t \in [0, T] \setminus D$ and $w \in \hat{\Omega}$, we have

$$\int_A f(t, \hat{X}_t, \hat{Y}_t, a)\hat{q}_t(da) = f(t, \hat{X}_t, \hat{Y}_t, \hat{U}_t),$$

$$\int_A h(t, \hat{X}_t, \hat{Y}_t, a)\hat{q}_t(da) = h(t, \hat{X}_t, \hat{Y}_t, \hat{U}_t)$$

and

$$\int_A b(t, \hat{X}_t, a)\hat{q}_t(da) = b(t, \hat{X}_t, \hat{U}_t).$$

Since $\hat{X}$ is continuous and $(\hat{Y}_\cdot, \int_0^\cdot \hat{Z}_s d\hat{W}_s)$ is cadlàg, then the process $(\hat{X}_t, \hat{Y}_t, \hat{Z}_t)$ satisfies, for each $t \in [0, T]$, the following system of FBSDEs

$$\begin{cases} \hat{X}_t = \hat{x} + \int_0^t b(s, \hat{X}_s, \hat{U}_s)ds + \int_0^t \sigma(s, \hat{X}_s)dW_s, \\ \hat{Y}_t = g(\hat{X}_T) + \int_t^T f(s, \hat{X}_s, \hat{Y}_s, \hat{U}_s)ds - \int_t^T \hat{Z}_s d\hat{W}_s - (\hat{N}_T - \hat{N}_t). \end{cases}$$

Moreover, $J(\hat{q}) = J(\hat{U})$.

## Acknowledgements

## Appendix. S-topology

The S-topology has been introduced by Jakubowski [20], as a topology defined on the Skorokhod space of càdlàg functions $\mathbb{D}([0, T]; \mathbb{R}^k)$. This topology is weaker than the Skorokhod topology and the tightness criteria are easier to establish. This criteria is the same as that of the Meyer & Zheng topology [22]. The topology $S$ arises naturally in limit theorems for stochastic integrals (see [24]) and has good continuity properties with respect to the Skorokhod problem for convex domains. We summarize some of its properties:

(1) If $x_n \to_S x_0$, then $x_n(t) \to_S x_0(t)$ for each $t$ except for a countable set.
(2) If $x_n(t) \to_S x_0(t)$ for each $t$ in a dense set containing 0 and $T$ and $\{x_n\}$ is $S$-relatively compact, then $x_n \to_S x_0$ (not true for the convergence in measure).
(3) We recall (see [22,11]) that for a family $(X^n)_n$ of quasi-martingales on the probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, the following condition ensures the tightness of the family $(X^n)_n$ on the space $\mathbb{D}([0, T]; \mathbb{R}^k)$ endowed with the $S$-topology

$$\sup_n \left( \sup_{0 \le t \le T} E|X_t^n| + CV(X^n) \right) < \infty,$$

where, for a quasi-martingale $X$ on $(\Omega, \{\mathcal{F}_t\}_{0 \le t \le T}, P)$, $CV(X)$ stands for the conditional variation of $X$ on $[0, T]$, and is defined by

$$CV(X) = \sup E\left( \sum_i |E(X_{t_{i+1}} - X_{t_i})/\mathcal{F}_{t_i}^n| \right),$$

where the supremum is taken over all partitions of $[0, T]$. Let $N^{a,b}(Y)$ denotes the number of up-crossing of the function $Y \in \mathbb{D}([0, T]; \mathbb{R}^k)$ in given levels $a < b$ (recall that $N^{a,b}(Y) \ge k$ if one can find numbers $0 \le t_1 < t_2 < \cdots < t_{2k-1} < t_{2k} \le T$ such that $Y(t_{2i-1}) < a$ and $Y(t_{2i}) > b$, $i = 1, 2, \ldots, k$).

**Lemma 1** (A Criteria for S-Tightness). *A sequence $(Y^n)_{n \in \mathbb{N}}$ is S-tight if and only if it is relatively compact on the S-topology. Let $(Y^n)_{n \in \mathbb{N}}$ be a family of stochastic processes in $\mathbb{D}([0, T]; \mathbb{R}^k)$. Then this family is tight for the S-topology if and only if $(\|Y^n\|_\infty)_n$ and $N^{a,b}(Y^n)$ are tight for each $a < b$.*

**Lemma 2** (The a.s. Skorokhod Representation). *Let $(\mathbb{D}, S)$ be a topological space on which there exists a countable family of S-continuous functions separating points in X. Let $\{X_n\}_{n \in \mathbb{N}}$ be a uniformly tight sequence of laws on $\mathbb{D}$. In every subsequence $\{X_{n_k}\}$ one can find a further subsequence $\{X_{n_{k_l}}\}$ and stochastic processes $\{Y_l\}$ defined on $([0, T], \mathcal{B}_{[0,T]}, l)$ such that*

$$Y_l \sim X_{n_{k_l}}, \quad l = 1, 2, \ldots \tag{1}$$

*for each $w \in [0, T]$*

$$Y_l(w) \underset{S}{\to} Y_0(w), \quad as\ l \to \infty, \tag{2}$$

*and for each $\varepsilon > 0$, there exists an S-compact subset $K_\varepsilon \subset \mathbb{D}$ such that*

$$P(\{w \in [0, T] : Y_l(w) \in K_\varepsilon, l = 1, 2, \ldots\}) > 1 - \varepsilon. \tag{3}$$

One can say that (2) and (3) describe "the almost sure convergence in compacts" and that (1)–(3) define the strong a.s. Skorokhod representation for subsequences ("strong" because of condition (3)).

**Remark 3.** The projection $\pi_T : y \in (\mathbb{D}([0, T]; \mathbb{R}), S) \longmapsto y(T)$ is continuous (see Remark 2.4, p. 8 in [20]), but $y \longmapsto y(t)$ is not continuous for each $0 \le t \le T$.

**Lemma 4.** *Let $(X^n, M^n)$ be a multidimensional process in $\mathbb{D}([0, T]; \mathbb{R}^p)(p \in \mathbb{R}^*)$ converging to $(Y, M)$ in the S-topology. Let $(\mathcal{F}_t^{X^n})_{t \ge 0}$ (resp. $(\mathcal{F}_t^X)_{t \ge 0}$) be the minimal complete admissible filtration for $X^n$ (resp. X). We assume that $\sup_n E[\sup_{0 \le t \le T} |M_t^n|^2] < C_T \forall T > 0$, $M^n$ is a $\mathcal{F}^{X^n}$-martingale and $M$ is a $\mathcal{F}^X$-adapted. Then $M$ is a $\mathcal{F}^X$-martingale.*

**Lemma 5.** *Let $(Y^n)_{n>0}$ be a sequence of processes converging weakly in $\mathbb{D}([0, T]; \mathbb{R}^p)$ to $Y$. We assume that $\sup_n E[\sup_{0 \le t \le T} |Y_t^n|^2] < +\infty$. Hence, for any $t \ge 0$, $E[\sup_{0 \le t \le T} |Y_t|^2] < +\infty$.*

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