Uncertain Takagi–Sugeno Fuzzy Systems Stabilization via Switching Control

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Abstract. This paper adresses the stabilization problem of uncertain T–S fuzzy models using switching control. The T–S fuzzy model is described by a set of uncertain linear systems where the local system uncertainty depends on the fulfillment degree of the corresponding rule. An optimization procedure is used to design local controllers such as to maximize the quadratic stability region of each closed loop local uncertain system. The local controllers design is based on the resolution of a set of LMIs and the global control law is obtained by switching between local controllers. A numerical simulation of the stabilization of the uncertain lorenz system is given to illustrate the efficiency of the proposed method.

Key words: TS fuzzy model, uncertain system, LMI, quadratic stability

Résumé. Dans cet article nous étudions le problème de stabilisation des modèles flous T–S incertains. Le modèle flou de Takagi–Sugeno incertain est décrit par un ensemble de systèmes linéaires incertains dont l'incertitude du système local est exprimée en fonction du poids de la règle correspondante. Une procedure d'optimisation est utilisée pour la synthèse des lois de commande locales de façon à maximiser la région de stabilité quadratique pour chaque modèle incertain local. La synthèse des lois de commande locales est basée sur la résolution d'un ensembles de LMIs et la loi de commande globale est obtenue par commutation entres les lois de commande locales. Un exemple de simulation est utilisé pour démontrer l'efficacité de la méthode proposée.

Mots-clés: Modèle flou T–S, système incertain,LMI, stabilité quadratique.

1 Introduction

In recent years, there has been growing interest in the study of the the Takagi–Sugeno (T–S) fuzzy system due to the fact that it provides a general framework to represent a non linear plant by using a set of local linear models [1]-[3]. A large number of systematic stability analysis and controller design results have been appeared in fuzzy control literature. Tanaka et al. discussed the stability and the design of fuzzy control systems in [4], they gave some checking conditions for stability, which can be used to design fuzzy control laws and several methods have been proposed to relax these stability conditions [5],[6],[7]. Robust stability has also been considered in [8]. Unfortunately, the stability conditions require the existence of a common positive definite matrix for all local linear models; this is a difficult problem to be solved in many cases, especially when the number of rules is large. Representation of fuzzy models by a set of linear uncertain systems has been suggested by Kim et al.[9], based on linear uncertain system theory several control design approaches has been proposed [9], [10],[11]. The drawback of the precedent approaches is that the LMIs or the algebraic Riccati equations used to check the stability may be infeasible and the number of local controllers is not optimized. In [12]–[13], a switching control design approach has been proposed to stabilize non linear systems via fuzzy models. It is based on the maximization of the quadratic stability region of each local model. The uncertainty of each fuzzy local model is represented in function of its fulfillment degree. To overcome the problem of infeasibility the fulfillment degree is incorporated in the LMIs. A maximization procedure is used to compute the minimal degree for which the LMI is feasible. In this paper, this approach is extended to non linear systems represented by uncertain T–S fuzzy models. The rest of the paper is organized as follows. Section 2 introduces the uncertain T–S fuzzy model and Section 3 presents the switching controller design approach for uncertain fuzzy dynamic models based on the maximization of quadratic stability region of each local model. To demonstrate the efficiency of the proposed approach, a simulation example is given in section 4. Finally, conclusions are given in section 5.

2 Uncertain Takagi-Sugeno Fuzzy Model

The uncertain continuous-time Takagi-Sugeno fuzzy dynamic model is a piecewise interpolation of several uncertain linear models through membership functions . The uncertain T–S fuzzy model is described by a set of fuzzy if-then rules. The i^{th} rule of the fuzzy model take the form:

Rule i: If $z_1(t)$ is $F_1^i, \cdots,$ and $z_g(t)$ is F_g^i

$$\text{Then } \begin{cases} \dot{\mathbf{x}}(t) = (\mathbf{A}_i + \Delta\mathbf{A}_i)\mathbf{x}(t) + (\mathbf{B}_i + \Delta\mathbf{B}_i)\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}_i\mathbf{x}(t) \end{cases} \quad (1)$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ denotes the state vector, $\mathbf{u}(t) \in \mathbb{R}^m$ the control vector, $\mathbf{y}(t) \in \mathbf{R}^p$ the output vector, F_j^i is the jth fuzzy set of the ith rule, $\mathbf{A}_i \in \mathbb{R}^{n\times n}, \mathbf{B}_i \in \mathbb{R}^{n\times m}$ and $\mathbf{C}_i \in \mathbb{R}^{p\times n}$ are the state matrix, the input matrix and the output matrix for the ith local model, r is the number of if-then rules, and $z_1(t), z_2(t) \cdots, z_g(t)$ are some measurable system variables. $\Delta\mathbf{A}_i \in \mathbb{R}^{n\times n}, \Delta\mathbf{B}_i \in \mathbb{R}^{n\times m}$ are unknown and possibly time-varying matrices representing the uncertainties in the system.

Using center-average defuzzification, product inference and singleton fuzzifier; the global dynamics of the fuzzy system can be described by

$$\begin{cases} \dot{\mathbf{x}}(t) = \sum_{i=1}^{r} \alpha_i(\mathbf{z}(t))\{(\mathbf{A}_i + \Delta\mathbf{A}_i)\,\mathbf{x}(t) + (\mathbf{B}_i + \Delta\mathbf{B}_i)\,\mathbf{u}(t)\} \\ \mathbf{y}(t) = \sum_{i=1}^{r} \alpha_i(\mathbf{z}(t))\mathbf{C}_i\mathbf{x}(t) \end{cases} \tag{2}$$

Where

$$\alpha_i(\mathbf{z}(t)) = \frac{\omega_i(\mathbf{z}(t))}{\sum_{i=1}^{r} \omega_i(\mathbf{z}(t))} \tag{3}$$

The scalars $\alpha_i(\mathbf{z}(t))$ are characterized by:

$$0 \leq \alpha_i(\mathbf{z}(t)) \leq 1 \text{ and } \sum_{i=1}^{r} \alpha_i(\mathbf{z}(t)) = 1 \tag{4}$$

We assume that the uncertain matrices $\Delta\mathbf{A}_i$ and $\Delta\mathbf{B}_i$ are norm bounded with the following structure

$$\left[\Delta\mathbf{A}_i \;\Delta\mathbf{B}_i\right] = \mathbf{D}_i \cdot \mathbf{F}_i(t) \cdot \left[\mathbf{E}_{i_A} \;\mathbf{E}_{i_B}\right] \quad i = 1, 2, ..., r \tag{5}$$

where $\mathbf{D}_i, \mathbf{E}_{i_A}$ and $\mathbf{E}_{i_B}$ are predetermined real constant matrices of appropriate dimensions, representing the structures of the system uncertainties. $\mathbf{F}_i(t)$ are unknown matrix-valued function satisfying

$$\mathbf{F}_i^T(t) \cdot \mathbf{F}_i(t) \leq \mathbf{I} \tag{6}$$

The T-S fuzzy model (2) has strong nonlinear interactions among its fuzzy rules which complicates the analysis and the control. In order to overcome these difficulties, the uncertain TS fuzzy model can be represented as a set of uncertain linear systems[10]. The global state space $\Omega \subseteq \mathbb{R}^n$ is partitioned into r subspaces, each subspace is defined as :

$$\Omega_l = \{\Omega \mid \alpha_l(\mathbf{z}(t)) > 0\} \tag{7}$$

These subspaces are characterized by:

$$\bigcup_{i=1}^{r} \Omega_i = \Omega \tag{8}$$

If the rules i and j can be inferred in the same time then :

$$\Omega_i \cap \Omega_j \neq \phi \tag{9}$$

If the rules i and j can't be inferred in the same time then :

$$\Omega_i \cap \Omega_j = \phi \tag{10}$$

In each subspace the uncertain TS fuzzy model (2) can be represented as:

$$\dot{\mathbf{x}}(t) = \left\{\alpha_l \tilde{\mathbf{A}}_l + (1-\alpha_l) \sum_{R_i \in \mathcal{R}_l}^{r} \alpha'_i(\mathbf{z}(t)) \tilde{\mathbf{A}}_i \right\} \mathbf{x}(t) + \left\{\alpha_l \tilde{\mathbf{B}}_l + (1-\alpha_l) \sum_{R_i \in \mathcal{R}_l}^{r} \alpha'_i(\mathbf{z}(t)) \tilde{\mathbf{B}}_i \right\} \mathbf{u}(t) \quad (11)$$

$$\mathbf{y}(t) = \left\{\alpha_l \mathbf{C}_l + (1-\alpha_l) \sum_{R_i \in \mathcal{R}_l}^{r} \alpha'_i(\mathbf{z}(t)) \mathbf{C}_i \right\} \mathbf{x}(t) \quad (12)$$

where

$$\tilde{\mathbf{A}}_i = \mathbf{A}_i + \Delta\mathbf{A}_i, \quad \tilde{\mathbf{B}}_i = \mathbf{B}_i + \Delta\mathbf{B}_i \quad (13)$$

and $\mathcal{R}_l$ is a rule subset containing rules that can be inferred in the same time as rule l.

$$\mathcal{R}_l = \{R_i, \exists t, \alpha_l(\mathbf{z}(t))\alpha_i(\mathbf{z}(t)) \neq 0\} \quad (14)$$

and

$$\alpha'_i(\mathbf{z}(t)) = \frac{\alpha_i(\mathbf{z}(t))}{1-\alpha_l(\mathbf{z}(t))} \quad (15)$$

The scalars $\alpha'_i(\mathbf{z}(t)),\ R_i \in \mathcal{R}_l$ are characterized by

$$0 \leq \alpha'_i(\mathbf{z}(t)) \leq 1, \qquad \sum_{R_i \in \mathcal{R}_l}^{r} \alpha'_i(\mathbf{z}(t)) = 1 \quad (16)$$

for $\alpha_l(\mathbf{z}(t)) = 1$ the uncertain fuzzy system can be represented by the uncertain linear local model corresponding to rule R_i.

3 Controller Design

We assume that the fuzzy system (2) is locally controllable, that is, the pairs $(\mathbf{A}_l, \mathbf{B}_l)$, $l = 1, \ldots, r$, are controllable. The basic idea is to design local feedback controllers by the maximization of the stability region of each closed loop local model. The switching controller consists of r_c linear state feedback controllers that will be switched from one to another to control the system. The switching controller can be described by:

$$\mathbf{u}(t) = \sum_{l \in \mathcal{I}_c} \zeta_l(\mathbf{x}(t)) \mathbf{u}_l(t) \quad (17)$$

with:

$$\mathbf{u}_l(t) = \mathbf{K}_l \mathbf{x}(t) \quad (18)$$

and:

$$\zeta_l(\mathbf{x}(t)) = \begin{cases} 1 & \mathbf{x}(t) \in \Omega_l^c \\ 0 & \text{otherwize} \end{cases} \quad (19)$$

where $\mathcal{I}_c \subseteq \mathcal{I} = \{1, 2, ..., r\}$ is a set containing the indexes of selected controllers, $\Omega_l^c \subseteq \Omega_l$ is the subregion in which the command is generated using the local state feedback $\mathbf{K}_l$ to be designed. It can be seen that (17) is a linear combination of r_c linear state feedback controllers, the number of controllers r_c may be different from the number of rules r. At each moment, only one of the linear state feedback controllers is chosen to generate the control signal. In [10]–[11] and [14] the number of controllers is the same as the number of rules while in this approach the number of necessary controllers to ensure the stabilization can be minimized and may be less then the number of rules.
The local controllers are designed so that the local stability region of each fuzzy subsystem is maximized. In the subregion Ω_l the control law is

$$\mathbf{u}(t) = \mathbf{u}_l(t) = \mathbf{K}_l\mathbf{x}(t), \quad \mathbf{x}(t) \in \Omega_l^c \tag{20}$$

where $\Omega_l^c \subseteq \Omega_l$ is the subregion of Ω_l where the fuzzy subsystem is stable using state feedback $\mathbf{K}_l$.

Theorem 1. *If there exist symmetric positive definite matrices* $\mathbf{X}_l = \mathbf{X}_l^T > 0, \mathbf{M}_l = \mathbf{M}_l^T > 0$ *positive scalars* $\varepsilon_{l_i} > 0, R_i \in \mathcal{R}_l$ *and scalar* $\mu_l > 0$ *such that*

$$\begin{bmatrix} \Phi_{ll} + \mathbf{M}_l & \varepsilon_{l_l}\mathbf{D}_l & [\mathbf{E}_{l_A}\mathbf{X}_l + \mathbf{E}_{l_B}\mathbf{Y}_l]^T \\ \varepsilon_{l_l}\mathbf{D}_l^T & -\varepsilon_{l_l}\mathbf{I} & \mathbf{0} \\ \mathbf{E}_{l_A}\mathbf{X}_l + \mathbf{E}_{l_B}\mathbf{Y}_l & \mathbf{0} & -\varepsilon_{l_l}\mathbf{I} \end{bmatrix} < 0 \tag{21}$$

$$\begin{bmatrix} \Phi_{li} - \mu_l\mathbf{M}_l & \varepsilon_{l_i}\mathbf{D}_i & [\mathbf{E}_{i_A}\mathbf{X}_l + \mathbf{E}_{i_B}\mathbf{Y}_l]^T \\ \varepsilon_{l_i}\mathbf{D}_i^T & -\varepsilon_{l_i}\mathbf{I} & \mathbf{0} \\ \mathbf{E}_{i_A}\mathbf{X}_l + \mathbf{E}_{i_B}\mathbf{Y}_l & \mathbf{0} & -\varepsilon_{l_i}\mathbf{I} \end{bmatrix} \le 0 \tag{22}$$

$$R_i \in \mathcal{R}_l, \ i \neq l$$

with

$$\Phi_{li} = \mathbf{X}_l\mathbf{A}_i^T + \mathbf{A}_i\mathbf{X}_l + \mathbf{B}_i\mathbf{Y}_l + \mathbf{Y}_l^T\mathbf{B}_i^T \tag{23}$$

then the fuzzy system is quadratically stablizable by the state feedback.

$$\mathbf{u}(t) = \mathbf{Y}_l\mathbf{X}_l^{-1}\mathbf{x}(t) \tag{24}$$

if

$$\alpha_l(\mathbf{z}(t)) \ge \frac{\mu_l}{1 + \mu_l}, \qquad \forall t \ge 0 \tag{25}$$

Proof. Consider the following Lyapunov function candidate

$$V = \mathbf{x}^T(t)\mathbf{P}_l\mathbf{x}(t) \tag{26}$$

$$\dot{V} = \mathbf{x}^T(t)\mathcal{L}(\mathbf{P}_l)\mathbf{x}(t) \tag{27}$$

where

$$\mathcal{L}(\mathbf{P}_l) = \alpha_l\mathcal{L}_l(\mathbf{P}_l) + (1 - \alpha_l)\sum_{R_i \in \mathcal{R}_l}^{r} \alpha_i'(\mathbf{z}(t))\mathcal{L}_i(\mathbf{P}_l) \tag{28}$$

$$\mathcal{L}(\mathbf{P}_l) = \sum_{R_i \in \mathcal{R}_l}^{r} \alpha_i'(\mathbf{z}(t)) \left[\alpha_l \mathcal{L}_l(\mathbf{P}_l) + (1-\alpha_l)\mathcal{L}_i(\mathbf{P}_l)\right] \tag{29}$$

$$\mathcal{L}_i(\mathbf{P}_l) = \left[\tilde{\mathbf{A}}_i + \tilde{\mathbf{B}}_i\mathbf{K}_l\right]^T \mathbf{P}_l + \mathbf{P}_l\left[\tilde{\mathbf{A}}_i + \tilde{\mathbf{B}}_i\mathbf{K}_l\right]$$

$$\begin{aligned}
\mathcal{L}_i(\mathbf{P}_l) =& \mathbf{A}_i^T\mathbf{P}_l + \mathbf{P}_l\mathbf{A}_i + \Delta\mathbf{A}_i^T\mathbf{P}_l + \mathbf{P}_l\Delta\mathbf{A}_i + \mathbf{K}_l^T\mathbf{B}_i^T\mathbf{P}_l + \mathbf{P}_l\mathbf{B}_i\mathbf{K}_l \\
&+ \Delta\mathbf{B}_i^T\mathbf{K}_l^T\mathbf{P}_l + \mathbf{P}_l\Delta\mathbf{B}_i\mathbf{K}_l \\
=& \left[\mathbf{A}_i + \mathbf{B}_i\mathbf{K}_l\right]^T\mathbf{P}_l + \mathbf{P}_l\left[\mathbf{A}_i + \mathbf{B}_i\mathbf{K}_l\right] + \left[\mathbf{E}_{i_A} + \mathbf{E}_{i_B}\mathbf{K}_l\right]^T\mathbf{F}_i^T(t)\mathbf{D}_i^T\mathbf{P}_l \\
&+ \mathbf{P}_l\mathbf{D}_i\mathbf{F}_i(t)\left[\mathbf{E}_{i_A} + \mathbf{E}_{i_B\mathbf{K}_l}\right] \\
\mathcal{L}_i(\mathbf{P}_l) \leq& \left[\mathbf{A}_i + \mathbf{B}_i\mathbf{K}_l\right]^T\mathbf{P}_l + \mathbf{P}_l\left[\mathbf{A}_i + \mathbf{B}_i\mathbf{K}_l\right] + \varepsilon_{l_i}\mathbf{P}_l\mathbf{D}_i\mathbf{D}_i^T\mathbf{P}_l \\
&+ \frac{1}{\varepsilon_{l_i}}\left[\mathbf{E}_{i_A} + \mathbf{E}_{i_B}\mathbf{K}_l\right]^T\mathbf{F}_i^T(t)\mathbf{F}_i(t)\left[\mathbf{E}_{i_A} + \mathbf{E}_{i_B}\mathbf{K}_l\right]
\end{aligned}$$

since

$$\mathbf{F}_{i_A}^T(t)\mathbf{F}_{i_A}(t) < \mathbf{I}$$

$$\begin{aligned}
\mathcal{L}_i(\mathbf{P}_l) \leq& \left[\mathbf{A}_i + \mathbf{B}_i\mathbf{K}_l\right]^T\mathbf{P}_l + \mathbf{P}_l\left[\mathbf{A}_i + \mathbf{B}_i\mathbf{K}_l\right] + \varepsilon_{l_i}\mathbf{P}_l\mathbf{D}_i\mathbf{D}_i^T\mathbf{P}_l \\
&+ \frac{1}{\varepsilon_{l_i}}\left[\mathbf{E}_{i_A} + \mathbf{E}_{i_B}\mathbf{K}_l\right]^T\left[\mathbf{E}_{i_A} + \mathbf{E}_{i_B}\mathbf{K}_l\right]
\end{aligned}$$

$$\mathcal{L}_l(\mathbf{P}_l) + \frac{1-\alpha_l}{\alpha_l}\mathcal{L}_i(\mathbf{P}_l) < 0 \Rightarrow \mathcal{L}(\mathbf{P}_l) < 0 \tag{30}$$

By multiplying the inequality by $\mathbf{X}_l = \mathbf{P}_l^{-1}$ on both sides we get

$$\begin{aligned}
&\mathbf{X}_l\mathbf{A}_l^T + \mathbf{Y}_l^T\mathbf{B}_l^T + \mathbf{A}_l\mathbf{X}_l + \mathbf{B}_l\mathbf{Y}_l + \varepsilon_{l_l}\mathbf{D}_l\mathbf{D}_l^T \\
&\quad + \frac{1}{\varepsilon_{l_l}}\left[\mathbf{E}_{l_A}\mathbf{X}_l + \mathbf{E}_{l_B}\mathbf{Y}_l\right]^T\left[\mathbf{E}_{l_A}\mathbf{X}_l + \mathbf{E}_{l_B}\mathbf{Y}_l\right] \\
&\quad + \frac{1-\alpha_l}{\alpha_l}\left\{\mathbf{X}_l\mathbf{A}_i^T + \mathbf{Y}_l^T\mathbf{B}_i^T + \mathbf{A}_i\mathbf{X}_l + \mathbf{B}_i\mathbf{Y}_l + \varepsilon_{l_i}\mathbf{D}_i\mathbf{D}_i^T\right. \\
&\quad \left. + \frac{1}{\varepsilon_{l_i}}\left[\mathbf{E}_{i_A}\mathbf{X}_l + \mathbf{E}_{i_B}\mathbf{Y}_l\right]^T\left[\mathbf{E}_{i_A}\mathbf{X}_l + \mathbf{E}_{i_B}\mathbf{Y}_l\right]\right\} < 0 \\
&\quad \Rightarrow \mathcal{L}(\mathbf{P}_l) < 0
\end{aligned}$$

Suppose that

$$\begin{aligned}
&\mathbf{X}_l\mathbf{A}_l^T + \mathbf{Y}_l^T\mathbf{B}_l^T + \mathbf{A}_l\mathbf{X}_l + \mathbf{B}_l\mathbf{Y}_l + \varepsilon_{l_l}\mathbf{D}_l\mathbf{D}_l^T \\
&\quad + \frac{1}{\varepsilon_{l_l}}\left[\mathbf{E}_{l_A}\mathbf{X}_l \; \mathbf{E}_{l_B}\mathbf{Y}_l\right]^T\left[\mathbf{E}_{l_A}\mathbf{X}_l \; \mathbf{E}_{l_B}\mathbf{Y}_l\right] + \mathbf{M}_l < 0
\end{aligned} \tag{31}$$

it yields

$$\frac{1-\alpha_l}{\alpha_l}\Big\{\mathbf{X}_l\mathbf{A}_i^T+\mathbf{Y}_l^T\mathbf{B}_i^T+\mathbf{A}_i\mathbf{X}_l+\mathbf{B}_i\mathbf{Y}_l+\varepsilon_{l_i}\mathbf{D}_i\mathbf{D}_i^T \\ +\frac{1}{\varepsilon_{l_i}}\left[\mathbf{E}_{i_A}\mathbf{X}_l+\mathbf{E}_{i_B}\mathbf{Y}_l\right]^T\left[\mathbf{E}_{i_A}\mathbf{X}_l+\mathbf{E}_{i_B}\mathbf{Y}_l\right]\Big\}\leq\mathbf{M}_l$$

$$\mathbf{X}_l\mathbf{A}_i^T+\mathbf{Y}_l^T\mathbf{B}_i^T+\mathbf{A}_i\mathbf{X}_l+\mathbf{B}_i\mathbf{Y}_l+\varepsilon_{l_i}\mathbf{D}_i\mathbf{D}_i^T \\ +\frac{1}{\varepsilon_{l_i}}\left[\mathbf{E}_{i_A}\mathbf{X}_l+\mathbf{E}_{i_B}\mathbf{Y}_l\right]^T\left[\mathbf{E}_{i_A}\mathbf{X}_l+\mathbf{E}_{i_B}\mathbf{Y}_l\right]-\mu_l\mathbf{M}_l\leq 0 \quad (32)$$

with

$$\mu_l=\frac{\alpha_l}{1-\alpha_l}\Rightarrow\alpha_l=\frac{\mu_l}{1+\mu_l} \quad (33)$$

By applying Schur Complement to (31) and (32) we obtain

$$\begin{bmatrix}\Phi_{ll}+\mathbf{M}_l & \varepsilon_{l_l}\mathbf{D}_l & \left[\mathbf{E}_{i_A}\mathbf{X}_l+\mathbf{E}_{i_B}\mathbf{Y}_l\right]^T \\ \varepsilon_{l_l}\mathbf{D}_l^T & -\varepsilon_{l_l}\mathbf{I} & \mathbf{0} \\ \mathbf{E}_{l_A}\mathbf{X}_l+\mathbf{E}_{l_B}\mathbf{Y}_l & \mathbf{0} & -\varepsilon_{l_l}\mathbf{I}\end{bmatrix}<0 \quad (34)$$

$$\begin{bmatrix}\Phi_{li}-\mu_l\mathbf{M}_l & \varepsilon_{l_i}\mathbf{D}_i & \left[\mathbf{E}_{i_A}\mathbf{X}_l+\mathbf{E}_{i_B}\mathbf{Y}_l\right]^T \\ \varepsilon_{l_i}\mathbf{D}_i^T & -\varepsilon_{l_i}\mathbf{I} & \mathbf{0} \\ \mathbf{E}_{i_A}\mathbf{X}_l+\mathbf{E}_{i_B}\mathbf{Y}_l & \mathbf{0} & -\varepsilon_{l_i}\mathbf{I}\end{bmatrix}\leq 0 \quad (35)$$

$$R_i\in\mathcal{R}_l,\ i\neq l$$

The quadratic stability region of the uncertain fuzzy sub-system (11) can be optimized by the following minimization program :

$$\begin{array}{l}\text{Minimize } \mu_l \\ \text{subject to } \mathbf{X}_l>0,\mathbf{M}_l>0,\quad \mu_l\geq 0,\varepsilon_{l_l}>0,\varepsilon_{l_i}>0 \\ \begin{bmatrix}\Phi_{ll}+\mathbf{M}_l & \varepsilon_{l_l}\mathbf{D}_l & \left[\mathbf{E}_{i_A}\mathbf{X}_l+\mathbf{E}_{i_B}\mathbf{Y}_l\right]^T \\ \varepsilon_{l_l}\mathbf{D}_l^T & -\varepsilon_{l_l}\mathbf{I} & \mathbf{0} \\ \mathbf{E}_{l_A}\mathbf{X}_l+\mathbf{E}_{l_B}\mathbf{Y}_l & \mathbf{0} & -\varepsilon_{l_l}\mathbf{I}\end{bmatrix}<0 \\ \begin{bmatrix}\Phi_{li}-\mu_l\mathbf{M}_l & \varepsilon_{l_i}\mathbf{D}_i & \left[\mathbf{E}_{i_A}\mathbf{X}_l+\mathbf{E}_{i_B}\mathbf{Y}_l\right]^T \\ \varepsilon_{l_i}\mathbf{D}_i^T & -\varepsilon_{l_i}\mathbf{I} & \mathbf{0} \\ \mathbf{E}_{i_A}\mathbf{X}_l+\mathbf{E}_{i_B}\mathbf{Y}_l & \mathbf{0} & -\varepsilon_{l_i}\mathbf{I}\end{bmatrix}\leq 0 \\ R_i\in\mathcal{R}_l,\ i\neq l\end{array} \quad (36)$$

with

$$\Phi_{li}=\mathbf{X}_l\mathbf{A}_i^T+\mathbf{A}_i\mathbf{X}_l+\mathbf{Y}_l^T\mathbf{B}_i^T+\mathbf{B}_i\mathbf{Y}_l \quad (37)$$

Let $\underline{\mu}_l$ be the solution of this minimization program, then the uncertain fuzzy subsystem (11) is quadratically stable if :

$$\forall t \quad \alpha_l(\mathbf{z}(t))\geq\underline{\alpha}_l=\frac{\underline{\mu}_l}{1+\underline{\mu}_l} \quad (38)$$

Let $\Omega \subseteq \mathbb{R}^n$ be the state space, we define the subregion Ω_l as:

$$\Omega_l = \{\Omega | \alpha_l(\mathbf{z}(t)) > 0\} \quad (39)$$

and $\Omega_l^s \subseteq \Omega_l$ as:

$$\Omega_l^s = \{\Omega_l | \alpha_l(\mathbf{z}(t)) \geq \underline{\alpha}_l\} \quad (40)$$

Definition 1. *We say that the stability covering condition[15] is satisfied if:*

$$\bigcup_{i=1}^{r} \Omega_l^s = \Omega \quad (41)$$

Lemma 1. *If there exists, at each moment t, at least one integer $k \in \{1, 2, \ldots, r\}$ such that :*

$$\alpha_k(\mathbf{z}(t)) \geq \underline{\alpha}_k \quad (42)$$

then the stability covering condition is fulfilled.

The resolution of the minimization program (36) for the r rules leads to three possible cases [13] :

- *Case 1.* Several or all $\underline{\alpha}_l = 0$, $l = 1, 2, \ldots, r$, the number of regions that verify the stability covering condition (41) may be less than the number of rules ($r_c < r$).
- *Case 2.* The number of necessary subregions to verify the stability covering condition (41) is the same as the number of rules ($r_c = r$).
- *Case 3.* The stability covering condition (41) is not fulfilled, the global system may be instable.

We define a new state space partition as:

$$\Omega_l^c = \{\Omega_l | \alpha_l(\mathbf{z}(t)) \geq \alpha_l^c\}, l \in \mathcal{I}_c \quad (43)$$

and

$$\bigcup_{I \in \mathcal{I}_c} \Omega_l^c = \Omega \quad (44)$$

where $\mathcal{I}_c \subseteq \mathcal{I} = \{1, 2, ..., r\}$ is a set containing the indexes of selected subregions to form the new state space partition and $\alpha_l^c \geq \underline{\alpha}_l$ define the boundary of each subregion Ω_l^c :

$$\partial \Omega_l^c = \{\Omega_l^c | \alpha_l(\mathbf{z}(t)) = \alpha_l^c\}, l \in \mathcal{I}_c \quad (45)$$

$\alpha_l^c, l \in \mathcal{I}_c$ are chosen such that the stability covering condition holds and any adjacents subregions Ω_i^c and Ω_j^c verify:

$$\Omega_i^c \cap \Omega_j^c = \partial \Omega_i^c \cap \partial \Omega_j^c \quad (46)$$

Theorem 2. *If there exist symmetric positive definite matrices $\mathbf{P}_l$, $\mathbf{Q}_l$ and positive scalars $0 \leq \alpha_l^c < 1, l \in \mathcal{I}_c$ such that:*

– *The stability covering condition is fulfilled:*

$$\bigcup_{l\in\mathcal{I}_c} \Omega_k^c = \Omega \tag{47}$$

–

$$\begin{bmatrix} \Phi_{ll}+\mathbf{M}_l & \varepsilon_{l_l}\mathbf{D}_l & [\mathbf{E}_{l_A}\mathbf{X}_l+\mathbf{E}_{l_B}\mathbf{Y}_l]^T \\ \varepsilon_{l_l}\mathbf{D}_l^T & -\varepsilon_{l_l}\mathbf{I} & \mathbf{0} \\ \mathbf{E}_{l_A}\mathbf{X}_l+\mathbf{E}_{l_B}\mathbf{Y}_l & \mathbf{0} & -\varepsilon_{l_l}\mathbf{I} \end{bmatrix} < 0 \tag{48}$$

$$\begin{bmatrix} \Phi_{li}-\mu_l\mathbf{M}_l & \varepsilon_{l_i}\mathbf{D}_i & [\mathbf{E}_{i_A}\mathbf{X}_l+\mathbf{E}_{i_B}\mathbf{Y}_l]^T \\ \varepsilon_{l_i}\mathbf{D}_i^T & -\varepsilon_{l_i}\mathbf{I} & \mathbf{0} \\ \mathbf{E}_{i_A}\mathbf{X}_l+\mathbf{E}_{i_B}\mathbf{Y}_l & \mathbf{0} & -\varepsilon_{l_i}\mathbf{I} \end{bmatrix} \leq 0 \tag{49}$$

$$R_i \in \mathcal{R}_l,\ i \neq l$$

with:

$$\mathbf{X}_l = \mathbf{P}_l^{-1}, \mathbf{M}_l = \mathbf{P}_l^{-1}\mathbf{Q}_l\mathbf{P}_l^{-1} \tag{50}$$

and

$$\alpha_l^c \geq \frac{\underline{\mu}_l}{1+\underline{\mu}_l} \tag{51}$$

–

$$\mathbf{P}_i \leq \mathbf{P}_j,$$

$$\text{for all transition states } \mathbf{x}(\tau^-) \in \Omega_j,\ \mathbf{x}(\tau^+) \in \Omega_i \tag{52}$$

then the fuzzy system (2) is globally stable.

Proof. Let the piecewise Lyapunov function candidate be defined by:

$$V(\mathbf{x}(t)) = \sum_{l\in\mathcal{I}_c} \zeta_i(\mathbf{x}(t))\mathbf{x}^T(t)\mathbf{P}_i\mathbf{x}(t) \tag{53}$$

with:

$$\zeta_i(\mathbf{x}(t)) = \begin{cases} 1\ \mathbf{x}(t) \in \Omega_i^c \\ 0 \text{ otherwize} \end{cases} \tag{54}$$

Let $\tau_k,\ k=1,\ldots,T$ be the time instants at which the state transit between two adjacent subregions. If at $t=\tau_k$ the state leaves the subregion Ω_i^c and enters into the subregion Ω_j^c then:

$$\mathbf{x}(\tau_k^-) \in \Omega_i^c, \quad \mathbf{x}(\tau_k^+) \in \Omega_j^c \tag{55}$$

We assume that there is no jump in the states.

$$\mathbf{x}(\tau_k^-) = \mathbf{x}(\tau_k^+) = \mathbf{x}(\tau_k), \quad k=1,\ldots,T \tag{56}$$

At the boundary states the Lyapunov function is defined by:

$$V(\mathbf{x}(\tau_k^-)) = \mathbf{x}^T(\tau_k)\mathbf{P}_i\mathbf{x}(\tau_k) \tag{57}$$

$$V(\mathbf{x}(\tau_k^+)) = \mathbf{x}^T(\tau_k)\mathbf{P}_j\mathbf{x}(\tau_k) \tag{58}$$

Since the stability covering condition is verified then :

$$\tau_k^+ < t < \tau_{k+1}^-, \quad k = 1, \dots, T, \quad \exists i \mid \mathbf{x}(t) \in \Omega_i \\ \Rightarrow \dot{V}(\mathbf{x}(t)) < 0, \quad \tau_k^+ < t < \tau_{k+1}^-, \quad k = 1, \dots, T \quad (59)$$

and condition (52) assures that:

$$V(\mathbf{x}(\tau_k^+)) < V(\mathbf{x}(\tau_k^-)), \quad k = 1, \dots, T \quad (60)$$

The Lyapunov function candidate is always decreasing and the fuzzy system is globally stable.

4 Simulation Example

In this example we simulate the control of an uncertain chaotic Lorenz system. The control objective is to drive its trajectory to the origin. The lorenz equations are as follows [14]:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -\sigma x_1(t) + \sigma x_2(t) \\ r x_1(t) - x_2(t) - x_1(t) x_3(t) \\ x_1(t) x_2(t) - b x_3(t) \end{bmatrix} \quad (61)$$

The nominal values of (σ, r, b) are $(10, 28, 8/3)$ for chaos to emerge. An exact fuzzy modeling is employed to construct a fuzzy model for the chaotic system, it utilizes the concept of sector nonlinearity [2]. The system (61) can be written in state representation as:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & -x_1(t) \\ 0 & x_1(t) & -b \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \quad (62)$$

Assume that $x_1(t) \in [M_1, M_2]$, then we can write :

$$x_1(t) = \frac{M_2 - x_1(t)}{M_2 - M_1} M_1 + \frac{x_1(t) - M_1}{M_2 - M_1} M_2 \quad (63)$$

and the the system (62) can written be as:

$$\dot{\mathbf{x}}(\mathbf{t}) = \omega_1(x_1(t)) \mathbf{A}_1 \mathbf{x}(t) + \omega_2(x_1(t)) \mathbf{A}_2 \mathbf{x}(t) \quad (64)$$

with $\mathbf{x}(t) = [x_1(t), x_2(t), x_3(t)]^T$ and :

$$\omega_1(x_1(t)) = \frac{M_2 - x_1(t)}{M_2 - M_1}, \quad \omega_2(x_1(t)) = \frac{x_1(t) - M_1}{M_2 - M_1}$$

$$\mathbf{A}_1 = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & -M_1 \\ 0 & M_1 & -b \end{bmatrix}, \ \mathbf{A}_2 = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & -M_2 \\ 0 & M_2 & -b \end{bmatrix}$$

By choosing $\omega_1(x_1(t))$ and $\omega_2(x_1(t))$ as membership functions, figure 1, the chaotic system can be exactly represented by the following Takagi-Sugeno fuzzy model:

$$\text{Rule } R^1: \text{ if } x_1(t) \text{ is about } M_1 \text{ Then } \dot{\mathbf{x}}(\mathbf{t}) = \mathbf{A}_1\mathbf{x}(t)$$
$$\text{Rule } R^2: \text{ if } x_1(t) \text{ is about } M_2 \text{ Then } \dot{\mathbf{x}}(\mathbf{t}) = \mathbf{A}_2\mathbf{x}(t)$$

We assume that the parameters (σ, r, b) are uncertain and vary around their

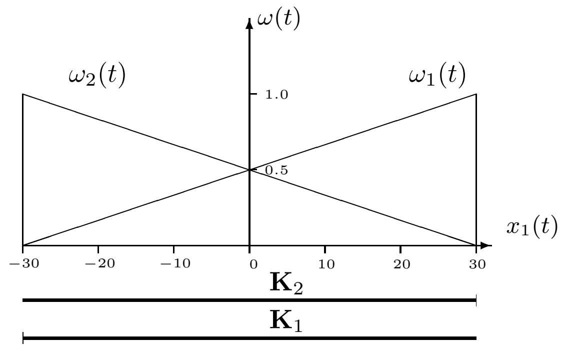

Fig. 1. Membership functions

nominal values $(\sigma_0, r_0, b_0) = (10, 28, 8/3)$ with:

$$\sigma = \sigma_0 + \Delta\sigma,\ r = r_0 + \Delta r,\ b = b_0 + \Delta b, \quad \Delta\sigma = \mu(t)\sigma_0,$$
$$\Delta r = \mu(t) r_0,\ \Delta b = \mu(t) b_0$$

We use the uncertain input matrix $\mathbf{B}$:

$$\mathbf{B} = \mathbf{B}_0 + \Delta\mathbf{B} \tag{65}$$

with:

$$\mathbf{B}_0 = [1, 0, 0]^T, \quad \Delta\mathbf{B} = \mu(t)\mathbf{B}_0 \tag{66}$$

and :

$$|\mu(t)| \leq 0.5 \tag{67}$$

The uncertain Lorenz chaotic system can be described by the following uncertain T-S fuzzy system:

$$R^1: \text{ If } x_1(t) \text{ is } M_1 \text{ Then } \dot{\mathbf{x}}(\mathbf{t}) = (\mathbf{A}_1 + \Delta\mathbf{A}_1)\mathbf{x}(t) + (\mathbf{B}_1 + \Delta\mathbf{B}_1)\mathbf{u}(t)$$
$$R^2: \text{ If } x_1(t) \text{ is } M_2 \text{ Then } \dot{\mathbf{x}}(\mathbf{t}) = (\mathbf{A}_2 + \Delta\mathbf{A}_2)\mathbf{x}(t) + (\mathbf{B}_2 + \Delta\mathbf{B}_2)\mathbf{u}(t)$$

with :

$$\Delta\mathbf{A}_1 = \Delta\mathbf{A}_2 = \begin{bmatrix} -\Delta\sigma & \Delta\sigma & 0 \\ \Delta r & 0 & 0 \\ 0 & 0 & -\Delta b \end{bmatrix}$$

$\Delta\mathbf{A}_1, \Delta\mathbf{A}_2, \Delta\mathbf{B}_1$ et $\Delta\mathbf{B}_2$ can be written as:

$$[\Delta\mathbf{A}_1\ \Delta\mathbf{B}_1] = \mathbf{D}_1 \cdot \mathbf{F}_1(t) \cdot [\mathbf{E}_{1_A}\ \mathbf{E}_{1_B}], \quad [\Delta\mathbf{A}_2\ \Delta\mathbf{B}_2] = \mathbf{D}_2 \cdot \mathbf{F}_2(t) \cdot [\mathbf{E}_{2_A}\ \mathbf{E}_{2_B}]$$

with:

$$\mathbf{D}_1 = \mathbf{D}_2 = 0.5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \mathbf{F}_1(t) = \mathbf{F}_2(t) = \mu'(t)\mathbf{I}_{3\times 3}$$

$$\mathbf{E}_{1_A} = \begin{bmatrix} -\sigma_0 & \sigma_0 & 0 \\ r_0 & 0 & 0 \\ 0 & 0 & -b_0 \end{bmatrix}, \quad \mathbf{E}_{2_A} = \begin{bmatrix} -\sigma_0 & \sigma_0 & 0 \\ r_0 & 0 & 0 \\ 0 & 0 & -b_0 \end{bmatrix}, \quad \mathbf{E}_{1_B} = \mathbf{E}_{2_B} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$$

and

$$\mu'(t) = 2.0\mu(t) \quad \Rightarrow \quad \begin{cases} \mathbf{F}_1(t) \cdot \mathbf{F}_1^T(t) \leq \mathbf{I} \\ \mathbf{F}_2(t) \cdot \mathbf{F}_2^T(t) \leq \mathbf{I} \end{cases}$$

The results obtained after the resolution of the minimisation program (36):

- *Subsystem 1:*

$$\underline{\alpha}_1 = 0, \quad \varepsilon_{1_1} = 4.8160, \quad \varepsilon_{1_2} = 3.5827$$

$$\mathbf{P}_1 = \begin{bmatrix} 17.0115 & 0.3839 & 0.0262 \\ 0.3839 & 0.3232 & 0.0017 \\ 0.0262 & 0.0017 & 0.3156 \end{bmatrix}, \quad \mathbf{K}_1 = \begin{bmatrix} -102.1530 & -12.9435 & -0.4785 \end{bmatrix}$$

- *Subsystem 2:*

$$\underline{\alpha}_2 = 0, \quad \varepsilon_{2_1} = 4.8160, \quad \varepsilon_{2_2} = 3.5827$$

$$\mathbf{P}_2 = \begin{bmatrix} 17.0115 & 0.3839 & -0.0262 \\ 0.3839 & 0.3232 & -0.0017 \\ -0.0262 & -0.0017 & 0.3156 \end{bmatrix}, \quad \mathbf{K}_2 = \begin{bmatrix} -102.1530 & -12.9435 & 0.4785 \end{bmatrix}$$

The boundary of the guaranteed stability subregions are determined by $\underline{\alpha}_1 = \underline{\alpha}_2 = 0$, figure 1, which means that $\Omega_1^s = \Omega_2^s = \Omega$ and the Lorenz chaotic system can be controlled using only one state feedback $\mathbf{u}(t) = \mathbf{K}_1\mathbf{x}(t)$ or $\mathbf{u}(t) = \mathbf{K}_2\mathbf{x}(t)$. The controller is able to drive the states to the origin for any initial conditions such that $x_1(t) \in [M_1, M_2]$. In figures 2- 4, the initial states are $\mathbf{x}(0) = [10, 10, 10]^T$ and the simulation time is 40 s.

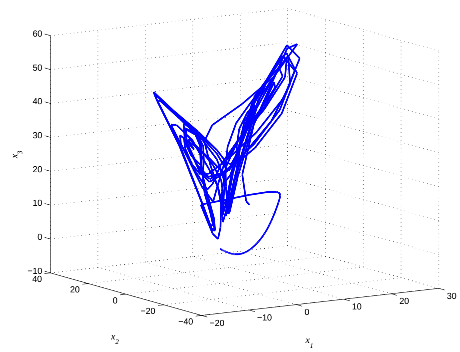

Fig. 2. Phase trajectory of the controlled uncertain Lorenz chaotic system

The control input is activated at $t = 20$ s using the linear state feedback $\mathbf{u}(t) = \mathbf{K}_1\mathbf{x}(t)$. Before the activation of the control the phase trajectory of the Lorenz system was chaotic. However, after the activation of the control the phase trajectory is quickly directed to the origin despite the time variation of the parameters (σ, r, b) of the original Lorenz system.

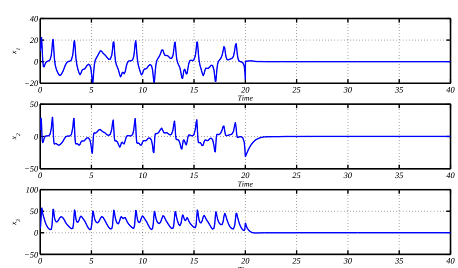

Fig. 3. States of the controlled uncertain Lorenz chaotic system

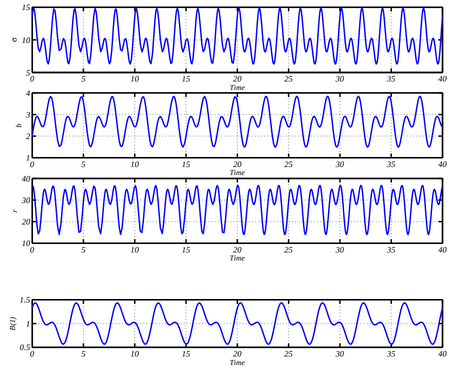

Fig. 4. Parameter variation of the uncertain Lorenz chaotic system

5 Conclusion

An LMI approach has been proposed to design a switching controller for uncertain T-S fuzzy models. The uncertain fuzzy model is represented as a set of uncertain linear systems and a local controller is designed such that the quadratic stability region of the corresponding local subsystem is maximized. This approach allows the optimization of the number of controllers which may be less than the number of rules. The stabilization of the uncertain Lorenz system has been used to demonstrate the effectiveness of this approach.

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