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Klein Paradox for the Bosonic Equation in the Presence of Minimal Length

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Abstract We present an exact solution of the one-dimensional modified Klein Gordon and Duffin Kemmer Petiau (for spins 0 and 1) equations with a step potential in the presence of minimal length in the uncertainty relation, where the expressions of the new transmission and reflection coefficients are determined for all cases. As an application, the Klein paradox in the presence of minimal length is discussed for all equations.

Keywords Klein Paradox · Klein Gordon · Duffin Kemmer Petiau equations · Minimal length

1 Introduction

Recently in quantum mechanics, there has been increasing interest in studying problems characterized by deformed commutation relations, including those concerning the minimal length uncertainty relation. The idea behind this deformation of the minimal length is to take into account the effects of quantum fluctuations of the gravitational field to integrate the theory of gravitation in quantum mechanics. One implication of this unification is the fact of introducing a minimum observable length, of the order of the Planck distance, in order to absorb the infinities vitiating the standard quantum theory. This minimal length uncertainty is seen as a distortion of the standard Heisenberg algebra by adding small corrections to the canonical commutation relations. This was

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motivated by string theory [1–6], quantum gravity [7], non-commutative geometry [8] and black hole physics [9,10].

If we look at the level of the nonrelativistic quantum mechanics, there has been several success in solving such problems. In this level, we cite for example the following potentials: the harmonic oscillator was solved exactly [11–19] and perturbatively [20], the Coulomb potential [21–25], the inverse square potential [26], the one-dimensional box [27], the Pauli-Hamiltonian [28] and finally the time-dependent linear potential [29].

But if we look to the relativistic domain, there was only few attempts to study the deformed theory by the presence of a minimal length. these works were on the generalized Dirac equation [30], the Dirac oscillator [31–33], the $(1+1)$-dimensional Klein–Gordon (KG) equation with mixed vector-scalar linear potentials [34] and the Duffin–Kemmer–Petiau (DKP) equation of the harmonic oscillator [35,36].

On the other hand, it is well known that the treatment of the diffused particle problem by a potential step or a barrier in the relativistic quantum mechanics leads us, when the potential is strong, to a paradoxical situation, called the Klein Paradox. This latter was first discussed by Oskar Klein in 1929 [37], when applying the Dirac equation to the electrons incident on a large potential step. At this stage, we note that, the study of this kind of problem for the other relativistic equations, by considering potentials with sharp boundaries, is very required since it enables us to cognize the existence of antiparticle and the qualitative explaining of phenomenons of particle pair creation. As the solution of the Dirac equation for high square barrier [38], Klein paradox of two-dimensional Dirac electrons in circular well potential in [39], KG equation with a step potential in minimal coupling [40], Klein paradox for bosons [41] and DKP equation with a square step potential [42] where the authors have shown that the Klein paradox does exist in the DKP theory.

The purpose of the present work is to solve exactly the one dimensional modified KG and DKP equations for spins 0 and 1 with a step potential, in order to study the Klein paradox in the bosonic theory with the presence of minimal length. We note that this problem was already discussed by Ghosh [43] in the case of the modified Dirac equation with a new kind of generalized uncertainty principle (GUP), where the energy band structures were determined to explain the nature of the Klein paradox.

The outline of this paper is as follows: In Sect. 2, we give some reviews of a minimal length relation. In Sect. 3, we expose an explicit calculation relative to the one-dimensional KG equation with a step potential in the context of minimal length, in the position space representation, where the α-dependent transmission and reflection coefficients are calculated. Then, the Klein paradox for spinless particles is resolved and its application shows that the strength of the potential can have a great effect on the existence of antiparticle and the explaining of phenomenon of pair production. Following the same study we determine in Sect. 4 the exact solutions of the modified DKP equation with the presence of the step potential in one-dimension relatively to particles of spin 1 and spin 0. In both cases, the exact solutions are obtained. Then, the Klein paradox which exists also for the deformed DKP theory is resolved. Some concluding remarks are given in the last section.

2 Review of a Minimal Length Relation

In the context of the deformed quantum mechanics on the basis of modified commutation relations between position and momentum operators, the usual Heisenberg algebra in one-dimensional case is renewed and changed to the so-called Generalized Uncertainty Principle (GUP) which obeys the following canonical commutation rules [44–47]

$$[X, P] = i\hbar\left(1 + \alpha p^2\right) \tag{1}$$

where α is a very small positive deforming parameter, which is generally expected to be of the order of the Planck length in quantum gravity $l_p \sim \sqrt{\frac{G\hbar}{c^3}} = 10^{-33} cm.$

According to algebra (1) which implements the minimal length, we have the deformed uncertainty relation which appears in perturbative string theory:

$$\Delta X \Delta P \geq \frac{\hbar}{2}\left[1 + \alpha\,(\Delta P)^2\right] \tag{2}$$

The relation (2) implies the appearance of a nonzero minimal length in positions $\Delta X_{\min} = \hbar\sqrt{\alpha}$.

From (1), the X and P are realized in momentum space as:

$$X = i\left(1 + \alpha p^2\right)\partial p, \qquad P = p \tag{3}$$

and in the position space, they act as:

$$X = x,\;\; P = -i\partial_x\left(1 - \tfrac{1}{3}\alpha\partial_x^2\right) \tag{4}$$

Over the rest of paper we adopt natural units such that $\hbar = c = 1$.

3 Klein Paradox for the KG Equation

In this section we consider a one dimensional KG equation with a step potential $V(x)$:

$$\left(P^2 + m^2\right)\Psi = (i\partial_t - V(x))^2\,\Psi \tag{5}$$

In order to treat the scattering of a KG particle of mass m and momentum P on the step potential $V(x)$ in the context of minimal length and in position space representation, we use (4) and we propose that the quantum fluctuations effects of the gravitational field are at order 1 in α.

$$\left(\tfrac{2}{3}\alpha\partial_x^4 - \partial_x^2 + m^2\right)\Psi = (i\partial_t - V(x))^2\,\Psi \tag{6}$$

from these equations, we have the following continuity equation:

$$\partial_t\rho_{KG} + \partial_x J_{KG} = 0 \tag{7}$$

The charge and the $1D$ current densities of the particle are defined as follows:

$$\rho_{KG} = \frac{i}{2m}\left[\Psi^{*}\left(\partial_t + iV\left(x\right)\right)\Psi - \Psi\left(\partial_t - iV\left(x\right)\right)\Psi^{*}\right], \quad J_{KG} = J_{0_{KG}} + \alpha J_{1_{KG}} \tag{8}$$

The current density J_{KG} is characterized by an usual quantum mechanical expression $J_{0_{KG}}$ and an additional α-dependent term correction, which is due to the influence of the space deformation:

$$J_{0_{KG}} = \frac{-i}{2m}\left[\Psi^{*}\partial_x\Psi - \Psi\partial_x\Psi^{*}\right] \tag{9a}$$

$$J_{1_{KG}} = \frac{i}{3m}\left[\left(\Psi^{*}\partial_x^3\Psi - \Psi\partial_x^3\Psi^{*}\right) - \left(\partial_x\Psi^{*}\partial_x^2\Psi - \partial_x\Psi\partial_x^2\Psi^{*}\right)\right] \tag{9b}$$

At this stage, the modified time-independent KG equation in the presence of the step potential $V\left(x\right)$ becomes in the regions $x < 0$ and $x > 0$ respectively as follows:

$$\left(-\tfrac{2}{3}\alpha\partial_x^4 + \partial_x^2 + k_{1,2}^2\right)\varphi_{1,2} = 0 \tag{10}$$

with (the anzats $\Psi\left(x,t\right) = \varphi\left(x\right) exp\left(-iEt\right)$ has been used):

$$V(x) = \begin{cases} 0 \ for \quad x < 0 \\ V_0 \ for \quad x > 0 \end{cases} \text{ and } k_1 = \sqrt{E^2 - m^2}, \quad k_2 = \sqrt{\left(E - V_0\right)^2 - m^2} \tag{11}$$

By a direct calculation, it is easy to obtain the following general solutions:

$$\varphi_1 = C_i e^{ik_i x} + C_r e^{-ik_i x} + C_{l_1} e^{k_{l_1} x} + C_{l_1}^{\grave{}} e^{-k_{l_1} x} \tag{12a}$$

$$\varphi_2 = C_t^{+} e^{ik_t x} + C_t^{-} e^{-ik_t x} + C_{l_2} e^{-k_{l_2} x} + C_{l_2}^{\grave{}} e^{k_{l_2} x} \tag{12b}$$

with $C_{i,r}$, $C_t^{\pm}$, C_{l_1,l_2} and $C_{l_1,l_2}^{\grave{}}$ are constant coefficients.

It is remarkable to note that the terms $e^{-k_{l_1} x}$ and $e^{k_{l_2} x}$ in (12a), (12b) becomes infinite respectively at $x \to \mp\infty$. Except for $C_{l_1,l_2}^{\grave{}} = 0$ (where the wave functions $C_{l_1,l_2}^{\grave{}} e^{\mp k_{l_1,l_2} x} \to 0$ are bounded). Consequently. Such a wave function which doesn't represent the scattering of a particle and none physical meaning doesn't belong to the solution (12a),(12b), the general solution becomes as follows

$$\varphi_1 = C_i e^{ik_i x} + C_r e^{-ik_i x} + C_{l_1} e^{k_{l_1} x} \tag{12c}$$

$$\varphi_2 = C_t^{+} e^{ik_t x} + C_t^{-} e^{-ik_t x} + C_{l_2} e^{-k_{l_2} x} \tag{12d}$$

Expressions of wave vectors contain, as it should be, α-corrections:

$$k_{i,t} = \sqrt{\frac{3}{4\alpha}\left(\sqrt{1 + \frac{8}{3}\alpha k_{1,2}^2} - 1\right)}, \quad k_{l_1,l_2} = \sqrt{\frac{3}{4\alpha}\left(\sqrt{1 + \frac{8}{3}\alpha k_{1,2}^2} + 1\right)} \tag{13a}$$

and can be simplified by a Taylor expansion to the leading order in α:

$$k_{i,t} = k_{1,2}\left(1 - \frac{\alpha}{3}k_{1,2}^2\right), \quad k_{l_1} = k_{l_2} = \sqrt{\frac{3}{2\alpha}} \tag{13b}$$

However, the shape of the wave function can be tested; using the limit $\alpha \to 0$ in the Eq. (12c),(12d), we obtain the following wave function of the scattering KG particle on the step potential $V(x)$ of ordinary quantum mechanics in position space representation

$$\varphi_1 = C_i e^{ik_1 x} + C_r e^{-ik_1 x} \tag{14a}$$

$$\varphi_2 = C_t^+ e^{ik_2 x} + C_t^- e^{-ik_2 x} \tag{14b}$$

In order to show the energetic bands allowing the propagative or non-propagative waves for this problem, we must know the nature of k_i with respect to energy E, based essentially on the study of the sign of values k_i^2 $\left(\text{or } k_t^2\right)$in the expression (13b) which can be divide the entire energy spectrum in five different regions as follows:

- For $k_i^2 = 0$ $\left(k_t^2 = 0\right)$ the particle encounters the standard forbidden band energy within the range $\pm m$ $(V_0 \pm m)$ and satisfies a new α-dependent limit values $E = \pm\sqrt{m^2 + \frac{3}{2\alpha}}$ $\left(E = V_0 \pm \sqrt{m^2 + \frac{3}{2\alpha}}\right)$ (the maximum positive and minimum negative energy).
- For $k_i^2 < 0$ $\left(k_t^2 < 0\right)$ suggests a imaginary k_i (or k_t)(damped mode) which corresponding a forbidden band according to the following three regions of the energy spectrum: $E < -\sqrt{m^2 + \frac{3}{2\alpha}}$ $\left(E < V_0 - \sqrt{m^2 + \frac{3}{2\alpha}}\right)$, $-m < E < m$ $(V_0 - m < E < V_0 + m)$ and $\sqrt{m^2 + \frac{3}{2\alpha}} < E$ $\left(V_0 - \sqrt{m^2 + \frac{3}{2\alpha}} < E\right)$.
- For $k_i^2 > 0$ $\left(k_t^2 > 0\right)$ suggests a real k_i (or k_t)(propagative mode) which corresponding a allowed band according to the following two regions of spectrum: $-\sqrt{m^2 + \frac{3}{2\alpha}} < E < -m$ $\left(V_0 - \sqrt{m^2 + \frac{3}{2\alpha}} < E < V_0 - m\right)$ and $m < E < \sqrt{m^2 + \frac{3}{2\alpha}}$ $\left(V_0 + m < E < V_0 + \sqrt{m^2 + \frac{3}{2\alpha}}\right)$.

In these bands energetic, one notes that apart from the standard forbidden band within the range $\pm m$, there are an additional forbidden bands which depends on deformation parameter α, that does not allow the scalar particles to penetrate the strong step potential region, we also note that this result is entirely similar to the result obtained by Ghosh in the study of the Klein paradox for Dirac particle with the presence of generalized uncertainty principle GUP which imply a minimum length and maximum energy[43].

Now, for the first region $x < 0$. When we choose the incoming particles with $E > 0$ ($k_i^2 > 0$), the solution $\varphi_1$contains an incoming plane wave $C_i e^{ik_i x}$ moving to the right with a positive group velocity $v_g = dE/dk_i = \frac{k_i}{E\left[1 - \frac{4\alpha}{3}(E^2 - m^2)\right]}$ which plays a crucial role in the classic description of the wave packet propagation, or energy (unlike to the case of the phase velocity), a reflected wave $C_r e^{-ik_i x}$ that goes to the left with a negative group velocity and an additional α-dependent term $C_{l_1} e^{k_{l_1} x}$ describing a damping mode; the latter is entirely due to the modification of the standard Heisenberg algebra.

In this region, the charge and current densities are calculated as follows:

$$\rho_{(i,r)KG} = \frac{E}{m}\left|C_{i,r}\right|^2, \; J_{\langle KG} = J_{iKG} + J_{rKG} = \frac{k_i}{m}\left(1 + \frac{4}{3}\alpha k_i^2\right)\left(|C_i|^2 - |C_r|^2\right) \tag{15}$$

For the second region $x > 0$, we assume that there is no reflected wave since the group velocity is the one of the moving wave packet (i.e. $v_g \geqslant 0$). Therefore the φ_2 solution must be an evanescent or a progressive wave according to the potential value V_0. Thence, one may distinguish four cases:

(a) for $m < V_0 < E - m$: k_t is real. If we choose $k_t > 0$, the group velocity is given by $v_g = dE/dk_t = \frac{k_t}{(E-V_0)\left[1-\frac{4\alpha}{3}\left((E-V_0)^2-m^2\right)\right]}$ and has the same direction as the wave vector k_t in this region. Thus the plane wave solution φ_2 has a positive direction (transmitted wave) only if $C_t^- = 0$. Charge and current densities take the forms:

$$\rho_{tKG}^{+} = \frac{(E - V_0)}{m}\left|C_t^+\right|^2, \; J_{tKG}^{+} = \frac{k_t}{m}\left(1 + \frac{4\alpha}{3}k_t^2\right)\left|C_t^+\right|^2 \tag{16}$$

(b) for $E - m < V_0 < E + m$: The wave number becomes purely imaginary $k_t = \pm i\,|k_t|$. Then, according to φ_2, there are exponentially decreasing solutions for the waves $C_t^+ e^{-|k_t|x}$ and $C_t^- e^{-|k_t|x}$ along x corresponding to a damping mode. At this point, the conditions $C_t^{\pm} = 0$ are necessary to avoid divergences in the current densities $J_t^{\pm}$ of the transmitted particle when $x \to \infty$.

$$\rho_{tKG}^{\pm} = \frac{(E - V_0)}{m}\left|C_t^{\pm}\right|^2 e^{-2|k_t|x}, \; J_{tKG}^{\pm} = 0 \tag{17}$$

(c) for $E+m < V_0 < E+\sqrt{m^2 + \frac{3}{4\alpha}}$: k_t assumes again real values. In this case, if we choose $k_t < 0$ the group velocity $v_g = \frac{k_t}{(V_0-E)\left[1-\frac{4\alpha}{3}\left((E-V_0)^2-m^2\right)\right]}$ has a positive sign and the solution φ_2 is a transmitted wave only if $C_t^+ = 0$. Charge and current densities takes the following negative values:

$$\rho_{tKG}^{-} = \frac{(E - V_0)}{m}\left|C_t^-\right|^2, \; J_{tKG}^{-} = -\frac{k_t}{m}\left(1 + \frac{4\alpha}{3}k_t^2\right)\left|C_t^-\right|^2 \tag{18}$$

(d) for $E + \sqrt{m^2 + \frac{3}{4\alpha}} < V_0 < \sqrt{m^2 + \frac{3}{2\alpha}}$, we have the same results as the case **a.**

In the case **c**, we note that the solution $C_t^- e^{-ik_t x}$ describes a propagate wave of a particle with a negative charge ρ_t^- in the positive direction, which is equivalent to the motion of a particle having an negative energy subjected to the action of a potential $-V_0$. Consequently, the solution can be interpreted as a propagation of an antiparticle that generates a negative charged current moving to the right of the x-axis.

At this stage, it is easy to calculate the transmission $T^{\pm}_{KG}$ and reflection R_{KG} coefficients by using the definition of the current densities:

$$T^{\pm}_{KG} = \left|\frac{J^{\pm}_t}{J_i}\right| = \frac{k_t}{k_i}\left(\frac{1+\frac{4\alpha}{3}k_t^2}{1+\frac{4\alpha}{3}k_i^2}\right)\left|\frac{C^{\pm}_t}{C_i}\right|^2, \quad R_{KG} = \left|\frac{J_r}{J_i}\right| = \left|\frac{C_r}{C_i}\right|^2 \tag{19}$$

On the other hand, the boundary conditions at $x = 0$ consist of four equations:

$$d^n\varphi_1 \,|_0 = d^n\varphi_2 \,|_0, \qquad n = 0, 1, 2, 3 \tag{20}$$

whose solutions are given for all previous cases as:

$$(\mathbf{a},\mathbf{d}) \begin{cases} \frac{C_r}{C_i} = \frac{k_i-k_t}{k_i+k_t} \\ \frac{C_t^+}{C_i} = \frac{2k_i}{k_i+k_t}\left(\frac{1+\frac{2}{3}\alpha k_i^2}{1+\frac{2}{3}\alpha k_t^2}\right) \\ \frac{C_t^-}{C_i} = 0 \end{cases}, \quad (\mathbf{b}) \begin{cases} \frac{C_r}{C_i} = \frac{(k_i - i|k_t|)^2}{k_i^2+k_t^2} \\ \frac{C_t^+}{C_i} = 0 \\ \frac{C_t^-}{C_i} = 0 \end{cases}, \quad (\mathbf{c}) \begin{cases} \frac{C_r}{C_i} = \frac{k_i+k_t}{k_i-k_t} \\ \frac{C_t^+}{C_i} = 0 \\ \frac{C_t^-}{C_i} = \frac{2k_i}{k_i-k_t}\left(\frac{1+\frac{2}{3}\alpha k_i^2}{1+\frac{2}{3}\alpha k_t^2}\right) \end{cases} \tag{21}$$

Transmission and reflection coefficients of the step potential can be easily reduced to:

$$(\mathbf{a},\mathbf{d}) \begin{cases} R_{KG} = \left|\frac{k_i-k_t}{k_i+k_t}\right|^2 = \left(\frac{k_1-k_2}{k_1+k_2}\right)^2\left(1-\frac{4\alpha}{3}k_1k_2\right) \\ T^-_{KG} = 0 \\ T^+_{KG} = \frac{4k_ik_t}{|k_i+k_t|^2} = \frac{4k_1k_2}{(k_1+k_2)^2}\left(1+\frac{\alpha}{3}(k_1-k_2)^2\right) \end{cases}, \quad (\mathbf{b}) \begin{cases} R_{KG} = \left|\frac{(k_i-i|k_t|)^2}{k_i^2+k_t^2}\right|^2 = 1 \\ T^-_{KG} = 0 \\ T^+_{KG} = 0 \end{cases},$$

$$(\mathbf{c}) \begin{cases} R_{KG} = \left|\frac{k_i+k_t}{k_i-k_t}\right|^2 = \left(\frac{k_1+k_2}{k_1-k_2}\right)^2\left(1+\frac{4\alpha}{3}k_1k_2\right) \\ T^-_{KG} = \frac{4k_ik_t}{|k_i-k_t|^2} = \frac{4k_1k_2}{(k_1-k_2)^2}\left(1+\frac{\alpha}{3}(k_1+k_2)^2\right) \\ T^+_{KG} = 0 \end{cases} \tag{22}$$

In **a**, **b** and**d**, we have $R_{KG} + T^+_{KG} = 1$, while we get $R_{KG} - T^-_{KG} = 1$ in **c**. The unitarity of this last case is still conserved, but only at the cost of a reflection coefficient exceeding unity ($R_{KG} > 1$). This shows that the number of reflected particles from the potential barrier is greater than that of the incident ones. Accordingly, we can conclude that the strong potential raises the GUP Klein paradox, by the existence of the phenomenon of pair production, from the threshold potential $V_0 = 2m$ at $x = 0$.

4 Klein Paradox for the DKP Equation

Before starting the study of the deformed DKP equation, let us expose some useful formulas. The one dimension DKP equation describing a scalar or a vector boson with a nonzero mass m in a step potential can be written as [48–56]:

$$\left[\beta^1 p+m\right]\Psi(x,t)=\beta^0\left(i\partial_0-V_0\right)\Psi(x,t) \tag{23}$$

where (β^1,β^0) are the DKP matrices and all their properties are listed in [48–50,53–56].

To derive the continuity equation in the context of minimal length, we write the equation in the x-representation with the use of (4):

$$\left[-i\beta^1\left(\partial_x-\frac{1}{3}\alpha\partial_x^3\right)+m\right]\Psi=\beta^0\left(i\partial_0-V_0\right)\Psi \tag{24}$$

From these equations, it is easy to define the adjoint spinor $\bar{\Psi}=\Psi^+\left(2\left(\beta^0\right)^2-1\right)$ and it verifies the following adjoint equation:

$$i\left(\partial_x-\frac{1}{3}\alpha\partial_x^3\right)\bar{\Psi}\beta^1+m\bar{\Psi}+\left(i\partial_0+V_0\right)\bar{\Psi}\beta^0=0 \tag{25}$$

In consequence, from Eqs. (24) and (25), we obtain the following expression:

$$\partial_0\left(\bar{\Psi}\beta^0\Psi\right)+\partial_x\left[\bar{\Psi}\beta^1\Psi-\frac{1}{3}\alpha\left(\partial_x^2\bar{\Psi}\beta^1\Psi+\bar{\Psi}\beta^1\partial_x^2\Psi-\partial_x\bar{\Psi}\beta^1\partial_x\Psi\right)\right]=0 \tag{26}$$

We note that the first and second terms in (26) represent the continuity equation of the usual one-dimensional DKP equation with a scalar potential and the third α-dependent term represents the correction due to the presence of the minimal length. This is similar to the case of non-commutative theory [57], where physical results depend on the space deformation parameter [58–60].

From these equations, we write the following continuity equation:

$$\partial_t\rho_{DKP}+\partial_x J_{DKP}=0 \tag{27}$$

where charge and $1D$ current densities of the DKP particle are defined as follows:

$$\rho_{DKP}=\bar{\Psi}\beta^0\Psi,\quad J_{DKP}=J_{0_{DKP}}+\alpha J_{1_{DKP}} \tag{28}$$

$$J_{0_{DKP}}=\bar{\Psi}\beta^1\Psi,\quad J_{1_{DKP}}=\frac{-1}{3}\left[\partial_x^2\bar{\Psi}\beta^1\Psi+\bar{\Psi}\beta^1\partial_x^2\Psi-\partial_x\bar{\Psi}\beta^1\partial_x\Psi\right] \tag{29}$$

It is remarkable to note that the time-component $J_{0_{DKP}}$ is not positive definite and may be interpreted as a charge density; it is positive for positive-energy states and negative for negative-energy ones [61].

In what follows, we use the $1D$ stationary equation describing a DKP particle in a step potential with the presence of minimal length (we put $\Psi(x,t)=e^{-iEt}\tilde{\Psi}(x)$):

$$\left[-i\beta^1\left(\partial_x-\frac{1}{3}\alpha\partial_x^3\right)+m\right]\tilde{\Psi}=\beta^0\left(E-V_0\right)\tilde{\Psi} \tag{30}$$

4.1 DKP Equation for Spin-One

In this section, it is obvious to note that the deformed DKP equation, as a relativistic equation, is fundamentally related to that of KG. Indeed, as we can see it, the equations of system (23) are not completely independent. The wave function $\tilde{\Psi}(x)^T$ has ten components (φ, **A**, **B**, **C**) where **A**, **B**, and **C** are vectors of dimension (3×1). It can be decomposed as: $\phi^T = (A_2, A_3, B_1)$, $\Phi^T = (B_2, B_3, A_1)$, $\Theta^T = (C_3, -C_2, \varphi)$ where A_i, B_i and C_i $(i = 1, 2, 3)$ are respectively the components of **A**, **B** and **C**.

Using the representation of DKP matrices (β^1, β^0) for the spin 1 case, the expressions of the charge and current densities are:

$$\rho_{DKP1} = 2\Re\left[\phi^+\Phi\right], \quad J_{DKP1} = -2\Re\left[\Theta^+\phi - \frac{\alpha}{3}\left(\partial_x^2\Theta^+\phi + \partial_x^2\phi^+\Theta - \partial_x\Theta^+\partial_x\phi\right)\right] \tag{31}$$

and according to (30), we obtain the following coupled system:

$$i\left(\partial_x - \tfrac{1}{3}\alpha\partial_x^3\right)\phi = m\Theta \tag{32a}$$

$$m\Phi = (E - V_0)\,\phi \tag{32b}$$

$$i\left(\partial_x - \tfrac{1}{3}\alpha\partial_x^3\right)\Theta + m\phi = (E - V_0)\quad\Phi \tag{32c}$$

The component C_1 automatically vanishes $(C_1 = 0)$.

As is well known that in the case of $(1 + 1)$ dimension, the relativistic DKP equation, describing the mixture of two bosonic sectors of spin 0 and 1 with a total absence of the spin effect is entirely equivalent to the KG equation which represents the fundamental and basic equation in the description of the relativistic quantum phenomenons [35, 62, 63], it is not difficult to verify that only $\phi(x)$ components of the predicted system are independents and can be reduced directly to the massive Klein–Gordon equation, which also represents the physical components of the DKP wave function for this kind of problem.

$$\left(-\tfrac{2}{3}\alpha\partial_x^4 + \partial_x^2 + (E - V_0)^2 - m^2\right)\phi\,(x) = 0 \tag{33}$$

The other components are determined by the following constraints equations:

$$\begin{pmatrix}\Phi\\ \Theta\end{pmatrix} = \frac{1}{m}\begin{pmatrix}(E - V_0)\\ i\left(\partial_x - \frac{1}{3}\alpha\partial_x^3\right)\end{pmatrix}\otimes\phi \tag{34}$$

Now, in the same way as for the KG case, we can obtain the following solutions:

$$\phi_< = \left[C_i^\backslash e^{ik_i x} + C_r^\backslash e^{-ik_i x} + C_{l_1}^\backslash e^{k_{l_1} x}\right]\mathbb{V} \qquad x < 0, \tag{35a}$$

$$\phi_> = \left[C_t^{\backslash +} e^{ik_t x} + C_t^{\backslash -} e^{-ik_t x} + C_{l_2}^\backslash e^{-k_{l_2} x}\right]\mathbb{V} \quad x > 0 \tag{35b}$$

where $\mathbb{V}$ is a constant vector of dimension (3×1) and with:

$$k_{i,t} = k_{1,2}\left(1 - \frac{\alpha}{3}k_{1,2}^2\right), \quad k_{l_1} = k_{l_2} = \sqrt{\frac{3}{2\alpha}} \tag{36a}$$

$$k_1^2 = E^2 - m^2, \quad k_2^2 = (E - V_0)^2 - m^2 \tag{36b}$$

By a direct calculation, it is easy to get the following final solutions:

$$\begin{pmatrix} \phi \\ \Phi \\ \Theta \end{pmatrix}_< = \left[\begin{pmatrix} 1 \\ \frac{E}{m} \\ -\frac{k_i}{m}\left(1 + \frac{\alpha}{3}k_i^2\right) \end{pmatrix} C_i^{\backprime} e^{ik_i x} + \begin{pmatrix} 1 \\ \frac{E}{m} \\ \frac{k_i}{m}\left(1 + \frac{\alpha}{3}k_i^2\right) \end{pmatrix} C_r^{\backprime} e^{-ik_i x} \right.$$
$$\left. + \begin{pmatrix} 0 \\ \frac{E}{m} \\ \frac{i}{2m}\sqrt{\frac{3}{2\alpha}} \end{pmatrix} C_{l_1}^{\backprime} e^{k_{l_1} x} \right] \otimes \mathbb{V} \tag{37}$$

$$\begin{pmatrix} \phi \\ \Phi \\ \Theta \end{pmatrix}_> = \frac{1}{\sqrt{6}} \left[\begin{pmatrix} 1 \\ \frac{(E - V_0)}{m} \\ -\frac{k_t}{m}\left(1 + \frac{\alpha}{3}k_t^2\right) \end{pmatrix} C_t^{\backprime +} e^{ik_t x} + \begin{pmatrix} 1 \\ \frac{(E - V_0)}{m} \\ \frac{k_t}{m}\left(1 + \frac{\alpha}{3}k_t^2\right) \end{pmatrix} C_t^{\backprime -} e^{-ik_t x} \right.$$
$$\left. + \begin{pmatrix} 0 \\ \frac{(E - V_0)}{m} \\ \frac{-i}{2m}\sqrt{\frac{3}{2\alpha}} \end{pmatrix} C_{l_2}^{\backprime} e^{-k_{l_1} x} \right] \otimes \mathbb{V} \tag{38}$$

where $C_{i,r}^{\backprime}$, $C_{l_1,l_2}^{\backprime}$ and $C_t^{\backprime\pm}$ are the normalization constants.

In this case, the charge and current densities can be calculated from the expression (31). For the first region $x < 0$, we have:

$$\begin{aligned} \rho_{(i,r)_{DKP1}} &= \frac{E}{m}\left|C_{i,r}^{\backprime}\right|^2, \quad J_{<DKP1} = J_{i_{DKP1}} + J_{r_{DKP1}} \\ &= \frac{k_i}{m}\left(1 + \tfrac{4}{3}\alpha k_i^2\right)\left(\left|C_i^{\backprime}\right|^2 - \left|C_r^{\backprime}\right|^2\right) \end{aligned} \tag{39}$$

For the second region $x > 0$, we found also four cases as in KG theory:

$$(\mathbf{a}, \mathbf{d}) \begin{cases} \rho_{t_{DKP1}}^{+} = \frac{(E-V_0)}{m}\left|C_t^{\backprime +}\right|^2 \\ J_{t_{DKP1}}^{+} = \frac{k_t}{m}\left(1 + \frac{4\alpha}{3}k_t^2\right)\left|C_t^{\backprime +}\right|^2 \end{cases}, \quad (\mathbf{b}) \begin{cases} \rho_{t_{DKP1}}^{\pm} = \frac{(E-V_0)}{m}\left|C_t^{\backprime \pm}\right|^2 e^{-2|k_t|x} \\ J_{t_{DKP1}}^{\pm} = 0 \end{cases},$$
$$(\mathbf{c}) \begin{cases} \rho_{t_{DKP1}}^{-} = \frac{(E-V_0)}{m}\left|C_t^{\backprime -}\right|^2 \\ J_{t_{DKP1}}^{-} = -\frac{k_t}{m}\left(1 + \frac{4\alpha}{3}k_t^2\right)\left|C_t^{\backprime -}\right|^2 \end{cases} \tag{40}$$

If we use the limit $\alpha \rightarrow 0$, we will find the exact result of the ordinary vector DKP equation with a step potential [53–56]:

$$\begin{pmatrix} \phi \\ \Phi \\ \Theta \end{pmatrix}_{<} = \left[\begin{pmatrix} 1 \\ \frac{E}{m} \\ -\frac{k_1}{m} \end{pmatrix} C_i^{\backslash} e^{ik_1 x} + \begin{pmatrix} 1 \\ \frac{E}{m} \\ \frac{k_1}{m} \end{pmatrix} C_r^{\backslash} e^{-ik_1 x} \right] \otimes \mathbb{V} \tag{41}$$

$$\begin{pmatrix} \phi \\ \Phi \\ \Theta \end{pmatrix}_{>} = \left[\begin{pmatrix} 1 \\ \frac{(E-V_0)}{m} \\ -\frac{k_2}{m} \end{pmatrix} C_t^{\backslash +} e^{ik_2 x} + \begin{pmatrix} 1 \\ \frac{(E-V_0)}{m} \\ \frac{k_2}{m} \end{pmatrix} C_t^{\backslash -} e^{-ik_2 x} \right] \otimes \mathbb{V} \tag{42}$$

4.2 DKP Equation for Spin-Zero

In this case, we proceed in the same way as in the case of spin 1 particle, by putting $\tilde{\Psi}(x)^T = (\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)$. With these components, the expressions of charge and current densities (28) (29) can be written as:

$$\rho_{DKP0} = 2\Re\left[\eta_1^* \eta_2\right], \quad J_{DKP0} = -2\Re\left[\eta_3^* \eta_1 - \frac{1}{3}\alpha\left(\partial_x^2 \eta_3^* \eta_1 + \partial_x^2 \eta_1^* \eta_3 - \partial_x \eta_3^* \partial_x \eta_1\right)\right] \tag{43}$$

and the equation system (32) is reduced to the following one with $\eta_4 = \eta_5 = 0$:

$$i\left(\partial_x - \frac{1}{3}\alpha\partial_x^3\right)\eta_1 = m\eta_3 \tag{44a}$$

$$m\eta_2 = (E - V_0)\,\eta_1 \tag{44b}$$

$$i\left(\partial_x - \frac{1}{3}\alpha\partial_x^3\right)\eta_3 + m\eta_1 = (E - V_0)\,\eta_2 \tag{44c}$$

Using the correspondences $\eta_1 \rightarrow \phi$, $\eta_2 \rightarrow \Phi$, $\eta_3 \rightarrow \Theta$ and $(\eta_4, \eta_5) \rightarrow C_1$, the solutions of the system (44) are deduced from those of spin 1 case (32):

$$\begin{pmatrix} \eta_2 \\ \eta_3 \end{pmatrix} = \frac{1}{m} \begin{pmatrix} (E - V_0) \\ i\left(\partial_x - \frac{\alpha}{3}\partial_x^3\right) \end{pmatrix} \otimes \eta_1 \tag{45}$$

with

$$\eta_{1<} = \left[C_i^{\backslash\backslash} e^{ik_i x} + C_r^{\backslash\backslash} e^{-ik_i x} + C_{l_1}^{\backslash\backslash} e^{k_{l_1} x}\right], \qquad x < 0 \tag{46a}$$

$$\eta_{1>} = \left[C_t^{\backslash\backslash +} e^{ik_t x} + C_t^{\backslash\backslash -} e^{-ik_t x} + C_{l_2}^{\backslash\backslash} e^{-k_{l_2} x}\right], \quad x > 0 \tag{46b}$$

where $C_{i,r}^{\backslash\backslash}$, $C_{l_1,l_2}^{\backslash\backslash}$ and $C_t^{\backslash\backslash\pm}$ are the normalization constants.

In the same way as before, we can arrive to the final results:

$$\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\end{pmatrix}_{<}=\left[\begin{pmatrix}1\\ \dfrac{E}{m}\\ -\frac{k_i}{m}\left(1+\frac{\alpha}{3}k_i^2\right)\end{pmatrix}C''_i e^{ik_i x}+\begin{pmatrix}1\\ \dfrac{E}{m}\\ \frac{k_i}{m}\left(1+\frac{\alpha}{3}k_i^2\right)\end{pmatrix}C''_r e^{-ik_i x}\right.$$
$$\left.+\begin{pmatrix}0\\ \dfrac{E}{m}\\ \dfrac{i}{2m}\sqrt{\frac{3}{2\alpha}}\end{pmatrix}C''_{l_1}e^{k_{l_1}x}\right] \tag{47}$$

$$\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\end{pmatrix}_{>}=\left[\begin{pmatrix}1\\ \dfrac{(E-V_0)}{m}\\ -\frac{k_t}{m}\left(1+\frac{\alpha}{3}k_t^2\right)\end{pmatrix}C''^{+}_t e^{ik_t x}+\begin{pmatrix}1\\ \dfrac{(E-V_0)}{m}\\ \frac{k_t}{m}\left(1+\frac{\alpha}{3}k_t^2\right)\end{pmatrix}C''^{-}_t e^{-ik_t x}\right.$$
$$\left.+\begin{pmatrix}0\\ \dfrac{(E-V_0)}{m}\\ \dfrac{-i}{2m}\sqrt{\frac{3}{2\alpha}}\end{pmatrix}C''_{l_2}e^{-k_{l_1}x}\right] \tag{48}$$

For the first region $x<0$, the charge and current densities are given by:

$$\rho_{(i,r)DKP0}=\frac{E}{m}\left|C''_{i,r}\right|^2,\ \ J_{<DKP0}=J_{iDKP0}+J_{rDKP0}=\frac{k_i}{m}\left(1+\frac{4}{3}\alpha k_i^2\right)\left(\left|C''_i\right|^2-\left|C''_r\right|^2\right) \tag{49}$$

And for the second region $x>0$, we get:

$$(\mathbf{a},\mathbf{d})\begin{cases}\rho^{+}_{tDKP0}=\frac{(E-V_0)}{m}\left|C''^{+}_t\right|^2\\ J^{+}_{tDKP0}=\frac{k_t}{m}\left(1+\frac{4\alpha}{3}k_t^2\right)\left|C''^{+}_t\right|^2\end{cases},\ (\mathbf{b})\begin{cases}\rho^{\pm}_{tDKP0}=\frac{(E-V_0)}{m}\left|C''^{\pm}_t\right|^2e^{-2|k_t|x}\\ J^{\pm}_{tDKP0}=0\end{cases},$$
$$(\mathbf{c})\begin{cases}\rho^{-}_{tDKP0}=\frac{(E-V_0)}{m}\left|C''^{-}_t\right|^2\\ J^{-}_{tDKP0}=-\frac{k_t}{m}\left(1+\frac{4\alpha}{3}k_t^2\right)\left|C''^{-}_t\right|^2\end{cases} \tag{50a}$$

We arrive now at the stage of determining transmission $\left(T^{\pm}_{DKP1},T^{\pm}_{DKP0}\right)$ and reflection (R_{DKP1},R_{DKP0}) coefficients of the step potential for both scalar and vector cases and we proceed in a similar manner as in the KG case; we obtain the following results:

$$(\mathbf{a},\mathbf{d})\begin{cases}R_{DKP1}=R_{DKP0}=\left(\frac{k_1-k_2}{k_1+k_2}\right)^2\left(1-\frac{4\alpha}{3}k_1k_2\right)\\ T^{-}_{DKP1}=T^{-}_{DKP0}=0\\ T^{+}_{DKP1}=T^{+}_{DKP0}=\frac{4k_1k_2}{(k_1+k_2)^2}\left(1+\frac{\alpha}{3}(k_1-k_2)^2\right)\\ \text{where } R_{DKP1}+T^{+}_{DKP1}=1,\quad R_{DKP0}+T^{+}_{DKP0}=1\end{cases},$$

$$
(\mathbf{b})\begin{cases} R_{DKP1}=R_{DKP0}=\left|\frac{(k_i-i|k_t|)^2}{k_i^2+k_t^2}\right|^2=1 \\ T^-_{DKP1}=T^-_{DKP0}=0 \\ T^+_{DKP1}=T^+_{DKP0}=0 \end{cases},
$$

$$
(\mathbf{c})\begin{cases} R_{DKP1}=R_{DKP0}=\left(\frac{k_1+k_2}{k_1-k_2}\right)^2\left(1+\frac{4\alpha}{3}k_1k_2\right) \\ T^-_{DKP1}=T^-_{DKP0}=\frac{4k_1k_2}{(k_1-k_2)^2}\left(1+\frac{\alpha}{3}\left(k_1+k_2\right)^2\right) \\ T^+_{DKP1}=T^+_{DKP0}=0 \\ \text{where } R_{DKP1}-T^-_{DKP1}=1,\quad R_{DKP0}-T^-_{DKP0}=1 \end{cases} \tag{51}
$$

From the unitarity of the case **c**, we have $R_{DKP(1,0)} > 1$ and we conclude that the Klein paradox exists also for the DKP theory deformed with a minimal length.

Once again if we use the limit $\alpha \to 0$, we recover the exact results of the scalar DKP equation for a spin 0 particle with a step potential without a minimal length [42,53–56]:

$$
\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\end{pmatrix}_< = \left[\begin{pmatrix}1\\ \frac{E}{m}\\ -\frac{k_1}{m}\end{pmatrix} C_i^{''} e^{ik_1x} + \begin{pmatrix}1\\ \frac{E}{m}\\ \frac{k_1}{m}\end{pmatrix} C_r^{''} e^{-ik_1x}\right] \tag{52}
$$

$$
\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\end{pmatrix}_> = \left[\begin{pmatrix}1\\ \frac{(E-V_0)}{m}\\ -\frac{k_2}{m}\end{pmatrix} C_t^{''+} e^{ik_2x} + \begin{pmatrix}1\\ \frac{(E-V_0)}{m}\\ \frac{k_2}{m}\end{pmatrix} C_t^{''-} e^{-ik_2x}\right] \tag{53}
$$

Before ending this section, we will study the Klein paradox for a massless vector DKP particle $m \to 0$, i.e., photons. Then, in order to exhibit how the Maxwell equations of photons which propagate along the positive x-direction appear as a zero-mass limit of the DKP equation. We use the wave function form $\tilde{\Psi}\,(x)^T$ [64] which is defined as $(-im\varphi, im\mathbf{A}, \mathbf{E}, -\mathbf{B})$ where $\mathbf{A}$ is vector of dimension (3×1),and $\mathbf{E}$, $\mathbf{B}$ represents respectively the electric and magnetic fields. It can be decomposed as: $\phi^T = (A_2, A_3, 0)$, $\Phi^T = (E_y, E_z, 0)$, $\Theta^T = (-B_z, B_y, 0)$ where A_i, E_i and B_i $(i = 1, 2, 3)$ are respectively the components of $\mathbf{A}$, $\mathbf{E}$ and $\mathbf{B}$.

Which leads us directly to the following equations system

$$
\left(\partial_x - \tfrac{1}{3}\alpha\partial_x^3\right)\phi = -\Theta \tag{54}
$$

$$
\Phi = i\,(\omega - V_0)\,\phi \tag{55}
$$

$$
\left(\partial_x - \tfrac{1}{3}\alpha\partial_x^3\right)\Theta = -i\,(\omega - V_0)\quad\Phi \tag{56}
$$

The component φ, A_1, E_x and B_x automatically vanishes. (where we have put $\Psi\,(x,t) = e^{-i\omega t}\tilde{\Psi}\,(x)$ with ω is the angular frequency)

Within this system, it is not difficult to verify that only $\Phi\,(x)$ components are independent and satisfy the following one-dimensional massless KG type equation:

$$\left(-\tfrac{2}{3}\alpha\partial_x^4+\partial_x^2+(\omega-V_0)^2\right)\phi(x)=0 \tag{57}$$

Now, in the same way as for the massive KG case, we can obtain the following solutions:

$$\phi_< = \left[C_i^{\backslash}e^{i\kappa_i x}+C_r^{\backslash}e^{-i\kappa_i x}+C_{l_1}^{\backslash}e^{\kappa_{l_1}x}\right]\mathbb{V} \qquad x<0, \tag{58}$$

$$\phi_> = \left[C_t^{\backslash+}e^{i\kappa_t x}+C_t^{\backslash-}e^{-i\kappa_t x}+C_{l_2}^{\backslash}e^{-\kappa_{l_2}x}\right]\mathbb{V} \qquad x>0 \tag{59}$$

And the other components of $\tilde{\Psi}(x)$ are determined by the following expressions:

$$\begin{pmatrix}\phi\\ \Phi\\ \Theta\end{pmatrix}_{\langle} = \left[\begin{pmatrix}1\\ i\omega\\ -i\kappa_i\left(1+\frac{\alpha}{3}\kappa_i^2\right)\end{pmatrix}C_i^{\backslash}e^{i\kappa_i x}+\begin{pmatrix}1\\ i\omega\\ i\kappa_i\left(1+\frac{\alpha}{3}\kappa_i^2\right)\end{pmatrix}C_r^{\backslash}e^{-i\kappa_i x}\right.$$
$$\left.+\begin{pmatrix}1\\ i\omega\\ \frac{-1}{2}\sqrt{\frac{3}{2\alpha}}\end{pmatrix}C_{l_1}^{\backslash}e^{\kappa_{l_1}x}\right]\otimes\mathbb{V} \tag{60}$$

$$\begin{pmatrix}\Phi\\ \Theta\\ \phi\end{pmatrix}_{\rangle} = \left[\begin{pmatrix}1\\ i(\omega-V_0)\\ -i\kappa_i\left(1+\frac{\alpha}{3}\kappa_i^2\right)\end{pmatrix}C_t^{\backslash+}e^{i\kappa_t x}+\begin{pmatrix}1\\ i(\omega-V_0)\\ i\kappa_i\left(1+\frac{\alpha}{3}\kappa_i^2\right)\end{pmatrix}C_t^{\backslash-}e^{-i\kappa_t x}\right.$$
$$\left.+\begin{pmatrix}1\\ i(\omega-V_0)\\ \frac{1}{2}\sqrt{\frac{3}{2\alpha}}\end{pmatrix}C_{l_2}^{\backslash}e^{\kappa_{l_1}x}\right]\otimes\mathbb{V} \tag{61}$$

where $\mathbb{V}$ is a constant vector of dimension (2×1) and:

$$\kappa_{i,t}=\kappa_{1,2}\left(1-\frac{\alpha}{3}\kappa_{1,2}^2\right),\quad \kappa_{l_1}=\kappa_{l_2}=\sqrt{\frac{3}{2\alpha}} \tag{62}$$

$$\kappa_1=\omega,\qquad \kappa_2=(\omega-V_0) \tag{63}$$

In the ordinary massless vector DKP equation, there is no forbidden energy band and the particle can penetrate the potential barrier for any height, but in the massless deformed case the number of energetic bands of preceding case is decreased to the three region as follows:

– For $k_i^2\,\langle 0$ (damped mode) which corresponding a forbidden band according to:

$$E<-\sqrt{\frac{3}{2\alpha}}\ \text{and}\ \sqrt{\frac{3}{2\alpha}}<E.$$

– For $k_i^2>0$ (propagative mode) which corresponding a allowed band according to:

$$-\sqrt{\frac{3}{2\alpha}} < E < \sqrt{\frac{3}{2\alpha}}$$

We note here, that the study of behavior of a massless vector DKP particle in inside a potential step in the deformed space, i.e., at high energy, showed that the new results of energetic bands that depends entirely on the deformation parameter α is similar to that of Ghosh in [43].

At this stage, we proceed in a similar manner as the massive vector particle DKP case, we obtain the following transmission $T^{\pm}_{MDKP1}$ and reflection R_{MDKP1} coefficients for the dynamics of a massless DKP spin 1 (MDKP1) particle in inside a potential step:

$$(\mathbf{a},\mathbf{d})\begin{cases} R_{MDKP1}=\left(\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\right)^2\left(1-\frac{4\alpha}{3}\kappa_1\kappa_2\right) \\ T^-_{MDKP1}=0 \\ T^+_{MDKP1}=\frac{4\kappa_1\kappa_2}{(\kappa_1+\kappa_2)^2}\left(1+\frac{\alpha}{3}\left(\kappa_1-\kappa_2\right)^2\right) \\ \text{where } R_{MDKP1}+T^+_{MDKP1}=1 \end{cases}, \qquad (\mathbf{b})\begin{cases} R_{MDKP1}=\left|\frac{(\kappa_i-i|\kappa_t|)^2}{\kappa_i^2+\kappa_t^2}\right|^2=1 \\ T^-_{MDKP1}=0 \\ T^+_{MDKP1}=0 \end{cases},$$

$$(\mathbf{c})\begin{cases} R_{MDKP1}=\left(\frac{\kappa_1+\kappa_2}{\kappa_1-\kappa_2}\right)^2\left(1+\frac{4\alpha}{3}\kappa_1\kappa_2\right) \\ T^-_{MDKP1}=\frac{4\kappa_1\kappa_2}{(\kappa_1-\kappa_2)^2}\left(1+\frac{\alpha}{3}\left(\kappa_1+\kappa_2\right)^2\right) \\ T^+_{MDKP1}=0 \\ \text{where } R_{MDKP1}-T^-_{MDKP1}=1 \end{cases} \tag{64}$$

From the case **c** of this result, we have $R_{MDKP1} > 1$, where we can concluded that the Klein paradox exists also for the deformed relativistic theory of the MDKP (photons) with the presence of a minimal length. We also note that this paradox exists in the ordinary case of MDKP contrary to the result obtained by Ghose et al via the Kemmer–Duffin–Chandra formalism [41].

5 Conclusion

In this paper, we have exposed an explicit calculation of the solutions in the position space representation of the one dimensional modified KG and DKP (spin 0 and 1) equations for the step potential in the presence of a minimal length. For both relativistic equations, we were obtained exact solutions and new expressions for the transmission and reflection coefficients with corrections depending on the deformation parameter α. This is not surprising since the modification of the Heisenberg algebra introduces new effects in physical results. The limiting case is then deduced when $\alpha \to 0$ and we have obtained, by using it, the results of ordinary relativistic quantum mechanics.

For the wave functions, in addition to corrections made to the usual solutions (these corrections are proportional to the parameter α as it should), we have shown that the deformation generates a new wave after interacting with the potential wall. This new solution describes an evanescent wave or a damping mode and is absent in the usual relativistic theory. It vanishes quickly because its depth is proportional to the deformation parameter and so, in view of the limits for this parameter from the literature, it is difficult to test it experimentally.

Looking at the side of the coefficients of transmission and reflection, we have also calculated the corrections to their expressions from the relativistic theory and this to the first order in the deformation parameter. We have demonstrated that these corrections preserve unitarity for the different relations in all cases considered, whether the KG theory or DKP theory for both spins 0 and 1. However, it is remarkable that in the presence of minimal length, the time-components of the conserved four-current $\rho^{-}_{t_{KG}}, \rho^{-}_{t_{DKP0}}, \rho^{-}_{t_{DKP1}}$ are not positive definite and can be interpreted as charge densities.

The fact that the α-corrections make no change to the emission relations (reflection and transmission) on the wall potential, but only modify them, lead us to conclude that the Klein paradox also exists in relativistic bosonic theories deformed by the minimum length and that our results are consistent with those of Cardoso et al. [42] in the DKP case and that they disapprove the statement of Ghose et al. [41].

Referring to the fact that the reflection coefficient, in all cases, is greater than unity for large values of the potential, we say that the so-called Klein paradox can be explained by considering that the potential, when it exceeds certain limits given by the theory considered, causes the phenomenon of pair creation and so the phenomenon should rather be called Klein pair creation (or production [65]) instead of Klein paradox.

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